Philosophical Magazine Series 3

ISSN: 1941-5966 (Print) 1941-5974 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm14

XLII. On the application of a new method to the geometry of curves and curve surfaces

J.W. Stubbs B.A.

low precipitates. When the lead salts were decomposed by sulphuretted hydrogen, I obtained an amorphous acid substance of a bright yellow colour, which was soluble in water, alcohol and æther, but which did not appear to be crystallizable.

XLII. *On the application of a new Method to the Geometry of Curves and Curve Surfaces.* *By* J. W. Stubbs, *B.A., Trinity College, Dublin.*

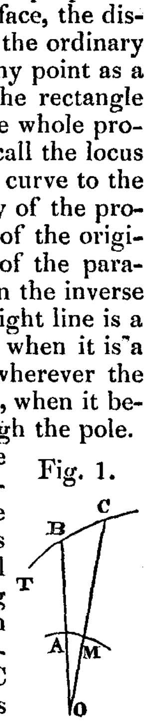
To the Editors of the Philosophical Magazine and Journal.

Gentlemen,

I HAD the honour of reading a paper before the Philosophical Society of Dublin, on a new Geometrical principle, which as far as I am aware has hitherto escaped the notice of mathematicians. May I ask of you the favour of inserting it in your valuable Journal?

The principle consists in taking the inverse of curves and surfaces, by means of which we readily find conjugate properties to those possessed by every known curve and surface, the discussion of many of which would be impossible by the ordinary methods. If in the plane of a curve we take any point as a pole and produce the radius vector, so that the rectangle under radius vector to the original curve and the whole produced radius be constant or equal to k^2, we may call the locus of the extremity of this produced line the inverse curve to the one from which it is produced, and the extremity of the produced radius the inverse point to the extremity of the original: as an example, the cardioide is the inverse of the parabola, the focus being the pole; the lemniscata in the inverse of the equilateral hyperbola. The inverse of a right line is a circle, except when the pole is on the right line, when it is a right line. The inverse of a circle is a circle wherever the pole is situated, except it be on the circumference, when it becomes a line perpendicular to the diameter through the pole.

Fig. 1.

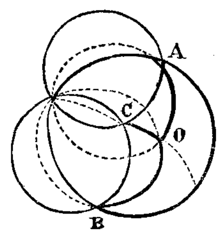


To draw a tangent to the inverse curve at the inverse point to a given point on the direct or generating curve, join the points, and on the joining line describe an isosceles triangle, one of whose sides is the tangent to the direct curve. The other will be the tangent to the inverse, as is seen by taking two consecutive radii; from the property by which it is generated the quadrilateral A M B C is circumscribable by a circle; hence the angle A M C equals the angle T B A, but in the limit the lines A M and B C become tangents: this is also clear from this, that $r\,r' = k^2$, then $r\,d\,r' = -\,r'\,d\,r$, and $\frac{r}{dr} = -\frac{r'}{dr'}$, or $\frac{r\,d\theta}{d\,r} = -\frac{r'\,d\theta}{d\,r'}$. Hence the perpendiculars from the pole on the tangents are as the radii, and the first perpendicular being known, the second is so also.

Having established these principles, I shall proceed to show the application of this method, first to the right line and circle, afterwards to curves of the second degree, and finally to surfaces.

If a circle passing through the pole be called a polar circle; from the known theorem of the bisectors of the angles of a triangle meeting in a point, by taking the inverse of all these lines we come to the following theorem: if three polar circles form by their intersection a polar triangle A B C, the polar circles A O, B O, C O bisecting the angles meet in a point O, which is the inverse of the point in which the original bisectors meet.

Fig. 2.



From the theorem of the three perpendiculars from the angles of a triangle on the opposite sides meeting in a point, we get by inversion the three polar circles perpendicular to the opposite sides of the polar triangle, and passing through the angles meet in a point.

In like manner every theorem in plane geometry, comprising only the right line and circle, gives a conjugate one, in which right lines and circles only are contained,—every theorem, I mean, which has relation only to *position*, without introducing *lengths of lines*. I shall not mention any more of them, as, when the principle is clearly seen, that to a line corresponds a circle, and to a point a point, to the contact of a line and circle, the contact of two circles or of a line and circle according as the pole is or is not on the circumference of the circle, and to the angle between two lines, the angle between the tangents to two circles at their point of intersection, any one can multiply theorems at will.

The general equation of a conic section being

$$A y^2 + B\,x y + C\,x^2 + D y + E\,x + F = 0,$$

substituting for y, $r \sin\theta$, and for x, $r\cos\theta$, we get

$$A\,r^2 \sin^2\theta + B\,r^2 \sin\theta \cos\theta + C\,r^2 \cos^2\theta + D\,r \sin\theta + E\,r \cos\theta + F = 0$$

the pole being at any point, put $r = \frac{k^2}{r'}$, or $\frac{1}{r'}$ for simplicity, and

$$\frac{A \sin^2 \theta}{r'^2} + \frac{B \sin \theta \cos \theta}{r'^2} + \frac{C \cos^2 \theta}{r'^2} + \frac{D \sin \theta}{r'} + \frac{E \cos \theta}{r'}$$
$$+ F = 0,$$

or multiplying by r^4,

$$A r'^2 \sin^2 \theta + B r'^2 \sin \theta \cos \theta + C r'^2 \cos^2 \theta + D r'^3 \sin \theta$$
$$+ E r'^3 \cos \theta + F r'^4 = 0$$

is the polar equation of the inverse conic section, its equation in rectangular coordinates being

$$A y^2 + B x y + C x^2 + D y (x^2 + y^2) + E x (x^2 + y^2)$$
$$+ (x^2 + y^2)^2 = 0.$$

1. If the focus be the pole, the distance from any point P to the focus is to its distance from the directrix in a constant ratio as e to 1.

Fig. 3.

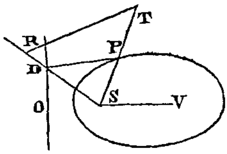
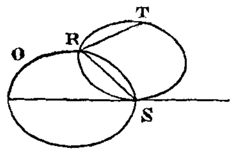


Fig. 4.



Now if we invert the line D O into a circle and the curve into the inverse focal ellipse whose equation is $r = k(1 - e \cos \omega)$, we can construct the focal inverse ellipse by a circular directrix; (in fig. 4) let S be the pole (which is the focus), S O any circle passing through S; (in fig. 3) produce S P to T so that $ST = \frac{1}{SP}$, and S D to R so that $RS = \frac{1}{SD}$ (which is the same as to invert the curve and directrix) from the similar triangles R T S, D P S R T : R S : : D P : P S : : 1 : e

$\because$ T R S = D P S = P S V.

Hence the circle circumscribing R S T is a tangent to S V at S; from this may be constructed the inverse focal conic section; for (in fig. 4) draw any chord S R to meet the circular directrix S R O, through S and R describe a circle S R T tangent to S V (the axis) at S, and inflect R T in a given ratio to R S, T is a point in the curve. As the cardioide is only a particular case of the focal conic section, this construction applies to it, making the ratio that of equality.

From the focal properties of conic sections we may deduce by inversion the following properties of the curve whose equation is $r = k(1 - e \cos \omega.)$.

In the parabola the perpendicular from the focus on the tangent meets it in the vertical tangent. Hence in the cardioide, if a polar circle be drawn tangent to the curve, the locus of the other extremity of the diameter passing through the pole is a circle passing through the cusp or pole and touching the curve at the opposite point, and consequently the locus of its centre is a circle.

In a conic section, if a point be taken inside the curve, and any chord be drawn, if we join the points in which it meets the curve with the focus, and also the given point with the focus, the product of the tangents of the half angles formed by those lines at the focus is constant; hence by inversion, if through a fixed point outside an inverse focal conic section we describe a polar circle, and join the points where it meets the curve with the pole, and also the given point with the pole, the product of the tangents of the half angles is constant.

In a conic section, if a chord subtend a constant angle at the focus, the envelope of the chord is a conic section with same focus and directrix; hence by inversion, if the arc of a polar circle contained between the points where it cuts the curve subtends a constant angle at the pole of a focal inverse conic, the envelope of this circle is an inverse focal conic with same pole and circular directrix.

If three tangents be drawn to a parabola, so as to form a triangle, the three angles and focus are in a circle; by inversion, if three circles be drawn through the pole of a cardioide touching the cardioide, the points of intersection are in a right line.

Every property, in fact, of a curve, with respect to any pole, has its analogous property in the inverse curve with respect to the same pole; to an asymptote in one, corresponds a circle passing through the pole and having its tangent at that point parallel to the asymptote, which the curve tends to approach as the radius diminishes; to a point of inflexion in one curve corresponds the property of the osculating circle at the conjugate or inverse point, passing through the pole; to a tangent in one corresponds a polar circle tangent to the other at the inverse point; to a cusp corresponds a cusp, and the osculating circle of the inverse curve is the inverse of that of the direct curve; so from the known properties of curves we can find the singular points of their inverse curves.

I shall not dwell any longer on those properties, as they are all obvious when the principle is explained. I shall merely show what Pascal's celebrated Theorem of the Hexagon inscribed in a conic section* becomes by inversion.

If in any inverse conic section six points be taken and six polar circles be described through each two consecutive points and the pole, the intersections of each opposite pair lie in a

[* See Phil. Mag. S. 3. vol. xxii. p. 168.—EDIT.]

circle passing through the pole; in the circle, the centre being the pole, this becomes a very remarkable theorem.

In two inverse curves the differential elements of the arcs are connected inthe following manner:— $d s' : d s :: r' : r$, or $d s' = \frac{k^4}{r^2} d s$; hence the differential element of the arc of a curve can be known when that of its inverse is known. This is remarkably connected with the theory of elliptic functions; the arc of an ellipse being represented by an elliptic function of the second kind, the arc of the curve formed by the intersections of the perpendiculars to the diameters of an ellipse through their extremities, by one of the first kind, I have found by this method that the arc of an inverse central ellipse is represented by $A\,\Pi\,(n, \phi) - B . F . \phi$, A and B being constants, and Π being an elliptic function of third kind with a circular parameter*, and $F\,\phi$ one of first kind; the amplitude ϕ being the angle by which the amplitude of the arc is measured in the ellipse from which it is generated; the x of the corresponding point of ellipse being $= a \sin \phi$, $y = b \cos \phi$. Hence from the general formula for the comparison of elliptic functions of the third kind (since if $\cos \sigma = \cos \phi \cos \psi - \sin \phi \sin \psi \sqrt{1 - c^2 \sin^2 \sigma}$ $F \phi + F \psi - F \sigma = 0$), viz.

$$\Pi(n\phi) + \Pi(n\psi) - \Pi(n\sigma) = \frac{1}{\sqrt{\alpha}} \tan^{-1} \frac{n \sqrt{\alpha} \sin \psi \sin \phi \sin \sigma}{1 + n - n \cos \psi \cos \phi \cos \sigma},$$

an infinite number of arcs of an inverse central ellipse may be found such that the difference between one measured from the vertex and the other between two other points shall be equal to a circular arc; and if the differenceof two arcs of an ellipse be equal to a right line, the difference of the arcs inverse to these shall be equal to a circular arc.

I will not trespass on your limits by proving this, it may be shown by the ordinary method; I will merely state the values of the constants; n the parameter $= \frac{a^2 - b^2}{b^2}$,

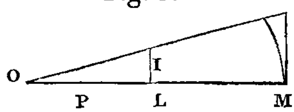
$A = \left(\frac{k^2 a}{b^2} + \frac{k^2 a c^2}{n b^2} \right)$, $B = \frac{k^2 c^2 a}{n b^2}$, a and b being the axes of the common ellipse, c its eccentricity, k the modulus of the inverse curve, or the constant equal to the rectangle under the coincident radii of the two curves. By discussing the formula above given for the comparison of elliptic functions of the third species, substituting for n and α $\left(\alpha \text{ is } (1+n)\,(1 + \frac{e^2}{n}) \right)$,

* This is not as general as I could wish, as there is a relation between the modulus and parameter which properly should be independent.

when the arcs are measured from the extremities of the axes, I come to the following theorem.

The difference of the arcs of an inverse ellipse, one measured from the end of the major, the other of the minor axis, and whose amplitudes fulfil the condition $b \tan \phi \tan \psi = a$ (a and b being the axes of the direct ellipse), is equal to the arc of a circle, which may be found by the following construction:—let I be the intercept between the foot of the perpendicular from the centre on the tangent of the direct ellipse and the point of contact, P the perpendicular from the centre on the line joining the extremities of the axes of the direct ellipse, and L the line joining the extremities of the axes of the inverse ellipse; then taking a line = P, raising at its extremity a perpendicular = I, and producing the line P until the whole line produced = L, with the common extremity O as centre, and L as radius, describe a circle, the arc of this circle intercepted between the other extremity, M, and the line joining O with the end of the perpendicular I, is the difference of the required arcs.

Fig. 5.



The analogy of this theorem to Fagnani's with regard to the direct ellipse (by which the difference of the corresponding arcs of it is I) is obvious. The area of the inverse ellipse is an arithmetic mean between the areas of the circles described on its axes.

To apply the inverse method to surfaces I will state the following principles: if one surface be inverse to another, a tangent plane being drawn at one point, the tangent plane at the inverse point is had by bisecting the line joining the points by a plane perpendicular to this line, and through the line where it cuts the tangent plane to the first surface, and the inverse point we draw a plane; it is a tangent plane at the inverse point: this is readily seen, as if through the common radius we draw any plane cutting the surfaces in two curves, these curves are inverse, and the construction which I gave for the tangents at inverse points makes this construction evident. Hence the normals at inverse points of surfaces are in the same plane and equally inclined to the common radius.

From this construction for the tangent plane, it follows that if two surfaces cut at right angles their inverse surfaces cut at right angles. Hence if we describe the developable surfaces formed by the tangent planes and normals at the points of a line of curvature, since these surfaces cut at right angles their inverse surfaces cut at right angles at the inverse points of the line of curvature, but the surface formed by the tangent planes to the inverse surface at those points is touched by the inverse of the corresponding surface in the first, and similarly the surface of normals by inverse of that in original surface; hence it may be seen that the normals to the inverse surface along the inverse points of a line of curvature meet consecutively, or the inverse of a line of curvature on a surface is the line of curvature of the inverse surface; or if the line of curvature of a surface be known, that of its inverse surface is had by describing a cone with the pole as vertex and passing through the line of curvature on direct surface, the line in which it pierces the inverse surface is a line of curvature.

Hence the umbilici of one surface correspond to the umbilici of the second; and in general to a tangent plane corresponds a polar sphere, and to all the singular points of one surface correspond singular points of the second.

From the known theorems of surfaces of the second order may be deduced numberless theorems of surfaces of the fourth order by inversion, similarly as in plane curves. I shall confine myself to the inverse of the central ellipsoid, which is Fresnel's surface of elasticity in the wave theory of light.

1. By the construction for the tangent plane to an inverse surface, the tangent plane to the surface of elasticity may be found from knowing the tangent plane of the ellipsoid.

2. The lines of curvature on that surface may be found by producing the cone passing through the lines of curvature on the ellipsoid to meet it, but

3. The intersection of a confocal ellipsoid and hyperboloid determines the line of curvature on either, as they cut at right angles; hence as the equation of the inverse ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(x^2+y^2+z^2)^2$, if two inverse central surfaces of the second order have their constants connected by the condition $a^2-a'^2=b^2-b'^2=c^2-c'^2$, and they intersect, they cut at right angles, and their intersection is the line of curvature on either.

4. By subtracting the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(x^2+y^2+z^2)^2, \text{ and}$$

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=(x^2+y^2+z)^2,$$

we get in the case above mentioned,

$$\frac{x^2}{a^2\,a'^2}+\frac{y^2}{b^2\,b'^2}+\frac{z^2}{c^2\,c'^2}=0,$$

the equation to a cone of the second order: the intersection of this cone with the surface inverse to the ellipsoid is the line of curvature.

5. By putting $x^2 + y^2 + z^2 =$ a constant, we find $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} =$ const., or the intersection of the surface of elasticity and concentric sphere is a spherical conic, since it is the same as the intersection of the surface of second order and sphere.

6. The circular sections of the surface of elasticity correspond to those of the ellipsoid, and the umbilici of either are found by the intersection of the diameters conjugate to the circular section of the ellipsoid, as at the umbilici the ultimate section must be a circle, and therefore parallel to the circular sections.

7. As to the rectilinear generatrices of the hyperboloid of one sheet correspond circular sections in the inverse hyperboloid, the latter has an infinite number of circular sections passing through the centre, but only two whose centre is at that point.

8. Hence the inverse hyperboloid of one sheet may be described by a moveable circle passing through a given point and moving on three others passing through the same point, which only cut in that point, and which neither lie in the same plane nor are circles of same sphere.

9. From the property that the sum of the squares of the reciprocals of three radii vectores at right angles to each other is constant in the ellipsoid, it follows that the sum of the squares of three rectangular semidiameters in the surface of elasticity is constant.

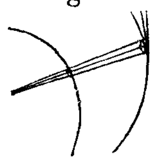
10. As the locus of the feet of perpendiculars from the centre on tangent planes of an ellipsoid is a surface of elasticity; by inversion, the locus of centres of spheres tangent to a surface of elasticity and passing through the centre is an ellipsoid.

11. From 8 and 5 it follows, that if planes pass through the centre of a surface of elasticity and cut out sections of a constant area, they envelope a cone of the second order, since the sum of the squares of the axes of the section is constant.

The foregoing are a few of the general theorems that may be deduced by the method I propose; they furnish a new instance of the duality that Chasles and others have remarked between the properties of figures; but it is superior to any hitherto proposed, as we can by it arrive at once at the properties of curves of higher orders which surpass our present power of analysis, from those of known curves; as from the known properties of curves of the second order, we come to those of the fourth with regard to polar circles, so by deducing those of the second order with regard to these latter, we might arrive at the properties of the higher curves with regard to lines. I shall not trespass on your limits any further by noticing new properties, many of which I have deduced in the paper before alluded to. I shall merely show the application of this method to physical investigation by two simple instances.

1. Since the ultimate elements of two inverse surfaces corresponding to each other are inversely placed with regard to the common radius, by describing an infinitesimal cone having these elements for their bases, $\frac{m}{d^2} = \frac{m'}{d'^2}$, m and m' being the masses of the bases, and d and d' the distances; hence the attraction of two infinitesimal elements resolved in any direction are the same, or the whole attractions of the corresponding parts of two inverse surfaces on the common pole is the same. The application of this to the plane and sphere is obvious.

Fig. 6.



2. The second theorem I shall state regards the wave-theory of light. It is stated by Sir John Herschel in his Essay on Light, that the equation for determining the velocity of a wave perpendicular to a given plane $z = m\,x + n\,y$, is

$$(V^2 - a^2)(V^2 - b^2) + m^2(V^2 - b^2)(V^2 - c^2) + n^2(V^2 - a^2)(V^2 - c^2) = 0,$$

and that this equation is had by an elimination which he states to be very complicated: it can be had at once by the following geometrical method.

Taking with him the equation of the surface of elasticity to be

$$R^4 = a^2 x^2 + b^2 y^2 + c^2 z^2;$$

if we find the intersection of this with the concentric sphere, we get

$$x^2(a^2 - r^2) + y^2(b^2 - r^2) + z^2(c^2 - r^2) = 0;$$

but if we put $r =$ to a constant, this represents a cone of second order. Now if we draw a tangent plane to this cone through the centre, the line of contact must be the axis of the curve cut out as the tangent to the sphere whose radius at r is perpendicular to it at its extremity; but this is also contained in the tangent plane, and therefore the intersection of these two planes is a tangent to the section perpendicular to its diameter, which is therefore an axis, but identifying the equation of the tangent plane to the cone, viz. $x\,x'(a^2 - r^2) + y\,y'(b^2 - r^2) + z\,z'(c^2 - r^2) = 0$, with the plane $l\,x + m\,y + n\,z = 0$, $l = x'(a^2 - r^2)$ $m = y'(b^2 - r^2)$, &c., $x'\,y'\,z'$ being the coordinates of the extremity of the diameter, but $x'^2 (a^2 - V^2) + y'^2 (b^2 - V^2) + z'^2 (c^2 - V^2) = 0$,

$$\therefore \frac{l^2}{(a^2 - V^2)} + \frac{m^2}{b^2 - V^2} + \frac{n^2}{c^2 - V^2} = 0,$$

which is the equation at which he arrives when we make $l = 1$.

I remain, Gentlemen, yours, &c.,

Trinity College, Dublin, Jan. 31, 1843.

JOHN WM. STUBBS.

XLIII. *On the Calorific Effects of Magneto-Electricity, and on the Mechanical Value of Heat.* By J. P. JOULE, *Esq.*

[Continued from p. 276.]

Part I. *On the Calorific Effects of Magneto-Electricity.*

I NOW proceeded to consider the heat evolved by voltaic currents when they are counteracted or assisted by magnetic induction. For this purpose it was only necessary to introduce a battery into the magneto-electrical circuit: then by turning the wheel in one direction I could oppose the voltaic current; or, by turning in the other direction, I could increase the intensity of the voltaic- by the assistance of the magneto-electricity. In the former case the apparatus possessed all the properties of the electro-magnetic engine; in the latter it presented the reverse, viz. the *expenditure* of mechanical power.

No. 7.

		Revolutions of Electro-Magnet per minute.	Deflections of Galvanometer of 1 turn.	Mean Temperature of Room.	Mean Difference.	Temperature of Water. Before.	Temperature of Water. After.	Loss or Gain.
May 19.	Circuits complete.	600	22° 40'	57°·43	1°·03−	55°·62	57°·18	1°·56 gain.
May 19.	Circuits broken.	600	0 0	57·45	0·41−	57·08	57·00	0·08 loss.
May 20.	Circuits complete.	600	20 45	59·40	0·08−	58·65	60·00	1·35 gain.
May 20.	Circuits broken.	600	0 0	59·40	0·51+	60·00	59·83	0·17 loss.
May 20.	Circuits complete.	600	23 0	59·10	1·29+	59·78	61·00	1·22 gain.
	Mean circuits complete.	600	22 8	....	0·06+	....	....	1·38 gain.
	Mean circuits broken.	600	....	....	0·05+	....	....	0·12 loss.
	Corrected Result.	600	22° 8' = 0·864 of current.					1·50 gain.