only an a priori determination, however uncertain, of the probability we are seeking. If we take the a priori probabilities ϖ for, and $(1-\varpi)$ against, instead of μ and ν, so that

$$p = \frac{m+\varpi}{m+n+1}, \tag{137}$$

then we are certain to avoid the paradox of unanimity where it might do harm, without deviating so much as the mean error from the observation in the a posteriori determination.

Neither Bayes's rule nor this latter one can be of any great use; but we can always employ them, when the found probabilities can be looked upon as definitive results. On the other hand, the formula of the mean value *may* be used in all cases, if we interpret the paradox of unanimity correctly. Where the found probabilities are to be subjected to adjustment, the latter formula, as I have said, *must* be employed; nor can the other rules be of any help in the cases where observed probabilities have to be rejected on account of the skewness of the law of errors.

XVII. MATHEMATICAL EXPECTATION AND ITS MEAN ERROR.

§ 74. Whether the theory of probability is employed in games, in insurances, or elsewhere, in all cases nearly in which we can speak of a favourable event, the prediction of the practical result is won through a computation of the mathematical expectation. The gain which a favourable event entails, has a value, and the chance of winning it must as a rule be bought by a stake. The question is: How are we to compare the value of the latter with that of which the game gives us expectation? Imagine the game to be repeated, and the number of repetitions N to become indefinitely large, then it is clear, according to the definition of probability, that the sum of the prizes won, if each of them is V, must be pNV, when p indicates the probability. The gain to be expected from every single game is consequently pV, and this product of the probability and the value of the prize is what we call mathematical expectation.

The adjective "mathematical" warns us not to consider pV as the real value which the possible gain has for a single player. This value, certainly, depends, not only objectively on the quantity of good things which form the prize, but also on purely subjective circumstances, among others on how much the player previously possesses and requires of the same sort of good things. An attempt which has been made to determine by means of what is called the "moral expectation", whether a game is advantageous or not, must certainly be regarded as a failure. For it takes into account the probable change in the logarithm of the player's property, but it does not take into consideration his requirements and other subordinate circumstances. We shall not here try to solve this difficulty.

It is evident, with respect to the mathematical expectation, that if we play several unbound games at the same time, the total mathematical expectation is equal to the sum of that of the several games. The same is the case, if we play a game in which each event entitles the player to a special (positive or negative) prize. In this latter case we speak of the total mathematical expectation as made up of partial ones.

Example 1. We play with a die in such a way that a throw of 1 or 2 or 3 wins nothing; a throw of 4 or 5 wins 2 s., and one of 6 wins 8 s. The total mathematical expectation is then $\frac{1}{2} \times 0 + \frac{1}{3} \times 2 + \frac{1}{6} \times 8 = 2$ s. A stake of 2 s. will consequently correspond to an even game. We might also deduce the 2 s. throughout, so that a throw of 1, or 2, or 3, causes a loss of 2 s. and a throw of 6 a gain of 6 s.; the total mathematical expectation then becomes $= 0$.

Example 2. In computations of the various kinds of life-insurances the basis is 1) the table of the number of persons $l(a)$ living at a given age a. The probability of such a person living x years is $= \frac{l(a+x)}{l(a)}$, of his dying within x years $= \frac{l(a) - l(a+x)}{l(a)}$, of his dying at the exact age of $a+x$ years $= -\frac{dl(a+x)}{l(a)\cdot dx}dx$, and from these all other necessary probabilities may be found; 2) the rate of interest ρ, which serves for the valuation of future payments of capital, $(1+\rho)^{-x}V$, or annuities certain $(1-(1+\rho)^{-x})\frac{v}{\rho}$.

The value of an endowment of capital, V, payable in x years, if the person who is now a years old is then alive, is thus equal to the mathematical expectation

$$V\frac{l(a+x)}{l(a)}(1+\rho)^{-x} = \frac{l(a+x)(1+\rho)^{-(a+x)}}{l(a)(1+\rho)^{-a}}V = \frac{D(a+x)}{D(a)}V, \tag{138}$$

which, as we see, is most easily computed by means of a table of the function

$$D(z) = l(z)(1+\rho)^{-z}.$$

Such a table is of great use for other purposes also.

The value of an annuity, v, due at the end of every year through which a person now a years old shall live, can be computed as a sum of such payments, or by the formula

$$v\sum_{x=1}^{x=\infty}\frac{l(a+x)}{l(a)}(1+\rho)^{-x} = \frac{v}{D(a)}\sum_{x=1}^{x=\infty}D(a+x), \tag{139}$$

where $l(\infty) = 0$ and $D(\infty) = 0$.

But it deserves to be mentioned that this same mathematical expectation is most safely looked upon as a total mathematical expectation in a game whose events are the various possible years of death; the probability of death in the first year being $\frac{l(a)-l(a+1)}{l(a)}$, in the second $\frac{l(a+1)-l(a+2)}{l(a)}$, and so on; while the corresponding values are annuities certain of v for varying duration. In this way we find for the value of the life-annuity the expression

$$\frac{v}{\rho\, l(a)} \sum_{x=0}^{x=\infty} (l(a+x) - l(a+x+1))(1-(1+\rho)^{-x}). \tag{140}$$

Since the sum $\sum_{0}^{\infty}(l(a+x)-l(a+x+1)) = l(a)$, we find by solution of the last parenthesis that the expression may be written

$$\frac{v}{\rho} - \frac{v}{\rho}\,\frac{1}{l(a)} \sum_{x=0}^{x=\infty} (l(a+x) - l(a+x+1))(1+\rho)^{-x},$$

and this shows that the value of the life-annuity is the difference between the capital sum of which the yearly interest is v and the value of a life-insurance of $\frac{v}{\rho}$ payable at the beginning of the year of death.

In life-insurance computations integrals are often employed with great advantage, instead of the sums we have used here; periodical payments (yearly, half-yearly, or quarterly) being reduced to continuous payments, and vice versâ.

§ 75. That mathematical expectation is not a solid value, but an uncertain claim, is expressed in the law of errors for the mathematical expectation, and particularly in its mean error; for, owing to the frequent repetitions and combinations in games and insurances, it does not matter much that the isolated laws of errors, here as for the probabilities, are often skew. If the value V is given free of error, the square of the mean error of the mathematical expectation, $H = pV$, is, according to the general rule, to be computed by

$$\lambda_2(H) = p(1-p)V^2. \tag{141}$$

If there are N repetitions of the same game we get

$$H' = pNV$$

and

$$\lambda_2(H') = p(1-p)NV^2; \tag{142}$$

and for the total expectation of mutually free games, $H'' = \Sigma p_i N_i V_i$, we have

$$\lambda_2(H'') = \Sigma p_i(1-p_i) N_i V_i^2. \tag{143}$$

By free games we may pretty safely understand such as are not settled by the various events of the same trial or game. (As to these, see § 76.)

The mean error is excellently adapted for computing whether we ought to enter upon a proposed game, or how highly we are to value uncertain claims or outstanding balance of accounts. Such things of course are regulated by the boldness or caution of the person concerned; but even the most cautious man may under fairly typical circumstances be contented with diminishing the value of his mathematical expectation by 4 times the amount of the mean error, and it would be sheer foolhardiness, if a passionate player would venture a stake which exceeded the mathematical expectation by the quadruple of its mean error. On the other hand, a simple subtraction or addition of the mean error cannot be counted a very strong proof of caution or boldness respectively.

Example 1. A game is arranged in such a way that the probability of winning from the person who keeps the bank is $\frac{1}{10}$, the prize is 8 \$. In n games the mathematical expectation with mean error is then $(0{\cdot}8\,n \pm 2{\cdot}4\sqrt{n})$ \$. If the banker has no property, but may expect 144 games to be played before the prizes are to be paid, he cannot without imprudence estimate his negative mathematical hope, his fear, lower than $0{\cdot}8 \times 144 + 2{\cdot}4 \times 12 = 144$ \$. He must consequently fix the stake for each game at about one dollar, and will thus stand a chance of seeing the bank broken about once in six times. If, however, he has got so much capital or credit, as also so many customers, that he can play about 2304 games, his business will become very safe; the average gain of 20 cts. per game is 460 \$ 80 cts., or exactly 4 times as great as the mean error 2 \$ 40 cts. $\times$ 48. But who will enter upon such a game against the banker (a game, after all, which is not worse than so many others)? The very stake is already greater than the mathematical expectation; every prudent regard to part of the mean error will only augment the disproportion. No prudent man will enter upon such a game, unless he can thereby avoid a greater risk: in this way we insure our risks, because it is too dangerous to be "one's own insurer". If the game is arranged in such an entertaining way that we pay 40 cts. for the excitement only of taking part in every game, then even rather a cautious person may also continue for 144 games, the mean error $(\pm 2{\cdot}4\sqrt{144}$ as above) being only 28 \$ 80 cts. or $144\,(0{\cdot}8 - (1{\cdot}0 - 0{\cdot}4))$ \$. For a poor fellow, who has only one dollar in his pocket, but who must for some reason necessarily get 8 \$, such a game may also be the best resource. But if a man owns only 2304 \$, and fails if he cannot get 8 times as much, then he would be exceedingly foolhardy if he played 2304 times or more in that bank. If we must run the risk, we can do no better than venturing everything on one card; if we distribute our chances over n repetitions, then we must, beyond the mathematical expectation, hope for $\sqrt{n}$ times that part of the mean error which might help by the one attempt.

Example 2. Two fire-insurance companies have each insured 10,000 farms for a total insurance of £ 10,000,000. The yearly probability of damage by fire is $\frac{1}{1000}$, and both must every year spend £ 5000 on management. Both have sufficient guaranty-fund to rest satisfied with one single mean error as security against a deficit in each fiscal year. How high must either fix its annual premium, when there is the difference that the company A has 10,000 risks of £ 1,000, while B has insured:

n		a	na	na^2
100	farms for	£ 10,000	£ 1,000,000	$10{,}000 \times (10)^6$
400	» » »	5,000	» 2,000,000	10,000 »
1,500	» » »	2,000	» 3,000,000	6,000 »
2,500	» » »	1,000	» 2,500,000	2,500 »
2,000	» » »	500	» 1,000,000	500 »
1,500	» » »	200	» 300,000	60 »
2,000	» » »	100	» 200,000	20 »
10,000	farms		£ 10,000,000	$29{,}080 \times (10)^6$

Since $p(1-p) = 0{\cdot}000999$, the mathematical expectation $\pm$ its mean error is in the case of A = £ 100,000 $\pm$ £ 3,161, in the case of B = £ 10,000 $\pm$ £ 5,390; the premiums are therefore £ 1 16 s. 4 d. and £ 2 7 s. 10 d. respectively for £ 1,000; *i. e.* B must reinsure part of its risks.

§ 76. The mean error and, in general, the law of error, of the total mathematical expectation for mutually bound events which may be considered co-ordinate events of the same trials, are computed in half-invariants by means of the sums of powers. If the trial can have n various events, of which the one whose probability is p_i entails a gain of the value a_i, and we imagine the same repeated a sufficiently large number of times (N times), the account will show:

$$a_1 \text{ occurring } p_1 N \text{ times,}$$

$$\cdots\cdots\cdots\cdots$$

$$a_n \text{ occurring } p_n N \text{ times.}$$

Hence

$$s_0 = (p_1 + \ldots + p_n) N$$
$$s_1 = (p_1 a_1 + \ldots + p_n a_n) N$$
$$s_2 = (p_1 a_1^2 + \ldots + p_n a_n^2) N,$$

and the half-invariants for the single trial will be

$$\lambda_1 = p_1 a_1 + \ldots + p_n a_n = \text{the total mathematical expectation} = H(1, \ldots n);$$
$$\lambda_2 (H(1, \ldots n)) = p_1 a_1^2 + \ldots + p_n a_n^2 - (p_1 a_1 + \ldots + p_n a_n)^2. \tag{144}$$

By this formula, therefore, we must in such cases compute the square of the mean error of the total mathematical expectation for the single trial. For the square of the mean error of the expectation from N trials we have consequently

$$\lambda_2 (N, H(1, \ldots n)) = N (p_1 a_1^2 + \ldots + p_n a_n^2 - H(1, \ldots n)^2). \tag{145}$$

By even game we understand a game where the total mathematical expectation is 0; the last term of this formula will consequently disappear in such a game. As the mean error does not depend on the absolute values of the gains or losses, but only on their differences, we may in the computation of the squares of the mean errors reduce to even game by subtracting the mathematical expectation from all the gains, and adding it to the losses. Thus we may write:

$$\lambda_2(N, H(1, \ldots n)) = N(p_1(a_1 - H(1, \ldots n))^2 + \ldots + p_n(a_n - H(1, \ldots n))^2). \quad (146)$$

This rule then differs from the rule of unbound games only in the absence of the factors $(1-p_1), \ldots (1-p_n)$.

We can now compute the mean errors in the examples 1 and 2, in § 74. In No. 1 we have

$$\lambda_2(H) = \tfrac{1}{3}(0)^2 + \tfrac{1}{3}(2)^2 + \tfrac{1}{3}(8)^2 - 2^2 =$$
$$= \tfrac{1}{3}(-2)^2 + \tfrac{1}{3}(0)^2 + \tfrac{1}{3}(6)^2 \quad = 8.$$

In the life-annuity example we now see the advantage of using the longer formula (140) for the value of the annuity, rather than the formula (139) which gives the value as the sum of a number of endowments; for the partial expectations are here not unbound, and only the deaths in the several years of age exclude one another and can be considered co-ordinate events in the same game. For the square of the mean error of the life-annuity we have, from (144):

$$\frac{v^2}{\rho^2 l(a)} \sum_{x=0}^{x=\infty} (l(a+x) - l(a+x+1))(1-(1+\rho)^{-x})^2 -$$
$$- \frac{v^2}{\rho^2 l(a)^2} \left\{ \sum_{x=0}^{x=\infty} (l(a+x) - l(a+x+1))(1-(1+\rho)^{-x}) \right\}^2. \quad (147)$$

§ 77. In the above studies on the mean errors of mathematical expectations we have supposed that the probabilities we use are free from error, being either determined a priori by good theory or found a posteriori from very large numbers of repetitions. This determination is not complete in the cases in which the probabilities determined a posteriori are found only by small numbers of trials, or if probabilities computed a priori presuppose values observed with sensibly large mean errors. The same warning must be taken with respect to other values which may enter into the computed mathematical expectations; the value of the gains, for instance, may depend on the future rate of interest. Whether some of the manifold sources of errors are to be omitted in a computation of the mean error, or not, must for each special case depend on the relative smallness of the parts of the total λ_2. As to the theory of probability it is characteristic only that the parts of the squares of the mean errors, considered in §§ 75 and 76, are, as a rule, very important, while the analogous parts in other problems are often insignificant. When the orbit of a planet is computed by the method of the least squares, then, in order to restrict the limits of research for its next discovery, we have to compute the mean errors of its co-ordinates at the next opposition. Ordinarily these mean errors are so large that the λ_2 for its future observations may be wholly omitted, though this λ_2 is analogous to those from §§ 75 and 76. But when we have computed a table of mortality by the method of the least squares, we can certainly find by that method the mean error $\sqrt{\lambda_2(p)}$ of the probability of life computed from the table; but if we are to predict anything as to the uncertainty with regard to n lives, and with regard to the corresponding mathematical expectation npa, then we must not, unless n is very great, take the mean error as $na\sqrt{\lambda_2(p)}$, but we must, as a rule, first take $\lambda_2(H)$ in consideration, and consequently use the formula $a\sqrt{np(1-p)+n^2\lambda_2(p)}$. (Comp. example, § 72).

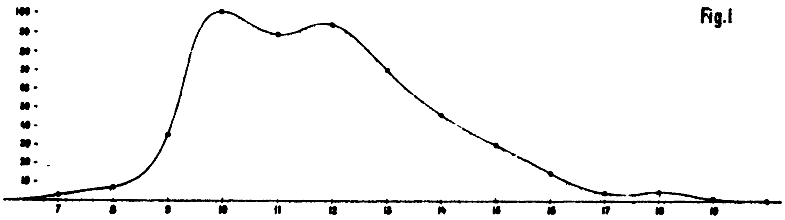

Fig.1

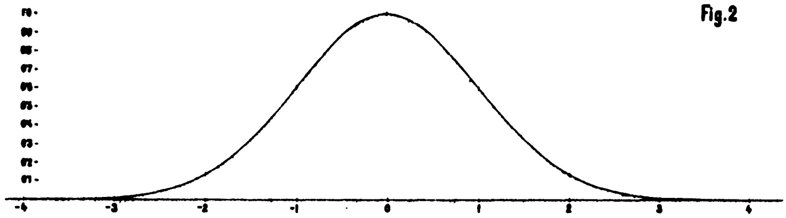

Fig.2

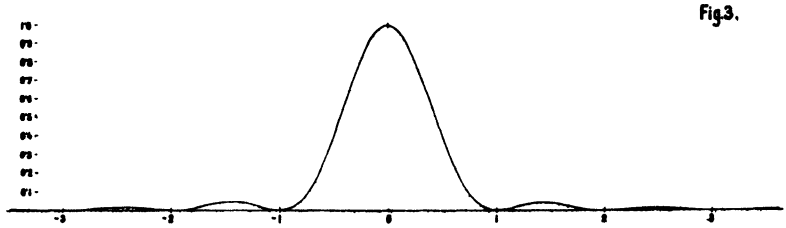

Fig.3.

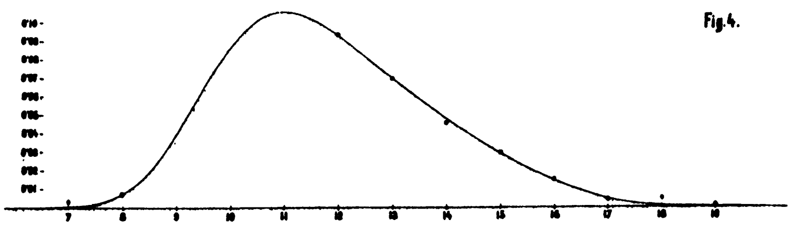

Fig.4.