XVIII.—On the Functions which are represented by the Expansions of the Interpolation-Theory. By E. T. Whittaker.

(MS. received May 14, 1915. Read June 7, 1915.)

§ 1. *Introduction.*

Let $f(x)$ be a given function of a variable x. We shall suppose that $f(x)$ is a one-valued *analytic* function, so that its Taylor's expansion in any part of the plane of the complex variable x can be derived from its Taylor's expansion in any other part of the plane by the process of analytic continuation.

Let the values of $f(x)$ which correspond to a set of equidistant values of the argument, say $a, a+w, a-w, a+2w, a-2w, a+3w, \ldots$ be denoted by $f_0, f_1, f_{-1}, f_2, f_{-2}, f_3$, etc. We shall suppose that these are all finite, even at infinity. Then denoting (f_1-f_0) by $\delta f_{\frac{1}{2}}$, (f_0-f_{-1}) by $\delta f_{-\frac{1}{2}}$, $(\delta f_{\frac{1}{2}}-\delta f_{-\frac{1}{2}})$ by $\delta^2 f_0$ etc., we can write out a "table of differences" for the function; the notation which will be used will be evident from the following scheme:—

$$
\left.\begin{array}{lllllll}
\text{Argument.} & \text{Entry.} & & & & & \\
a-2w & f_{-2} & \cdots & & & & \\
 & & \delta f_{-\frac{3}{2}} & \cdots & & & \\
a-w & f_{-1} & & \delta^2 f_{-1} & \cdots & & \\
 & & \delta f_{-\frac{1}{2}} & & \delta^3 f_{-\frac{1}{2}} \cdots & & \\
a & f_0 & & \delta^2 f_0 & & \delta^4 f_0 \cdots & \\
 & & \delta f_{\frac{1}{2}} & & \delta^3 f_{\frac{1}{2}} \cdots & & \\
a+w & f_1 & & \delta^2 f_1 \cdots & & & \\
 & & \delta f_{\frac{3}{2}} \cdots & & & & \\
a+2w & f_2 \cdots & & & & &
\end{array}\right\} \quad (1)
$$

Now it is obvious that $f(x)$ is not the only analytic function which can give rise to the difference-table (1): for we can form a new function by adding to $f(x)$ any analytic function which vanishes for the values $a, a+w, a-w, a+2w, \ldots$ of the argument, and this new function will have precisely the same difference-table as $f(x)$. All the analytic functions which give rise in this way to the same difference-table will be said to be *cotabular*. Any two cotabular functions are equal to each other when the argument has any one of the values $a, a+w, a-w, a+2w, \ldots$, but they are not equal to each other in general when the argument has a value not included in this set.

In the theory of interpolation certain expansions are introduced in order to represent the function $f(x)$, for general values of x, in terms of the quantities occurring in the above difference-table. We shall consider in particular the expansion

$$f_0+n\delta f_{\frac{1}{2}}+\frac{n(n-1)}{2\,!}\delta^2 f_0+\frac{(n+1)n(n-1)}{3\,!}\delta^3 f_{\frac{1}{2}}+\frac{(n+1)n(n-1)(n-2)}{4\,!}\delta^4 f_0+\ldots \quad (2)$$

which is supposed (when it converges) to represent $f(a+nw)$, where n can have any value. It is obvious, however, that there is no reason *a priori* why this expansion should represent $f(x)$ in preference to any other function of the set cotabular with $f(x)$: and thus two questions arise, namely :—

(1) *Which one of the functions of the cotabular set is represented by the expansion* (2)*?*

(2) *Given any one function* f(x) *belonging to the cotabular set, is it possible to construct from* f(x), *by analytical processes, that function of the cotabular set which is represented by the expansion* (2)*?*

These questions are answered in the present paper. It is, in fact, shown that there is a certain function belonging to the cotabular set which is represented by the expansion (2). This function is named the *cardinal function* of the set, and its properties are investigated. A formula is given by which the cardinal function may be constructed when any one function of the cotabular set is known.

§ 2. *Removal of singularities from a function, by substituting a cotabular function for it.*

We shall first show that *if* f(x) *has a singularity at a point* c, *we can find a function cotabular with* f(x) *which has no singularity at* c.

For suppose first that the singularity is a simple pole, so that $f(x)$ becomes infinite in the same way as

$$\frac{r}{x-c}$$

near the point c. Then the function

$$f(x)-\frac{r\sin\dfrac{\pi(x-a)}{w}}{(x-c)\sin\dfrac{\pi(c-a)}{w}}$$

is cotabular with $f(x)$, since the factor $\sin\dfrac{\pi(x-a)}{w}$ vanishes at all the places a, $a+w$, $a-w$, etc.: and this function has no singularity at c, since the infinite part of the term

$$-\frac{r\sin\dfrac{\pi(x-a)}{w}}{(x-c)\sin\dfrac{\pi(c-a)}{w}}$$

exactly neutralises the infinite part of $f(x)$. Moreover, this term does not introduce any fresh singularity in the finite part of the x-plane, and does not cause the new function to become infinite even at $x=\infty$ so long as x is real.

This establishes the result for the case when the singularity is a simple pole. When it is a pole of higher order, or an essential singularity, we can make use of the known result that the part of the expansion of $f(x)$ which becomes infinite near this singularity may be expressed in the form

$$-\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-x}$$

where γ denotes a small circle enclosing the singularity c. Now this can be neutralised by a term

$$\frac{1}{2\pi i}\sin\frac{\pi(x-a)}{w}\int_\gamma \frac{f(z)dz}{(z-x)\sin\dfrac{\pi(z-a)}{w}};$$

and as this term contains $\sin\dfrac{\pi(x-a)}{w}$ as a factor, it vanishes when the argument has any of the values a, $a+w$, $a-w$, $a+2w$, ... Hence in this case also we can write down a function, namely,

$$f(x)+\frac{1}{2\pi i}\sin\frac{\pi(x-a)}{w}\int_\gamma \frac{f(z)dz}{(z-x)\sin\dfrac{\pi(z-a)}{w}},$$

which is cotabular with $f(x)$ but has no singularity at the point $x=c$.

By repeated application of this process *we can remove all the singularities of* f(x) *in the finite part of the plane, and obtain a function which is cotabular with* f(x), *and which does not become infinite except for values of* x *whose imaginary part is infinite.*

§ 3. *Removal of rapid oscillations from a function, by substituting a cotabular function for it.*

Having replaced the original function $f(x)$ by a cotabular function of the kind just described, we shall now suppose the latter function to be analysed into periodic constituents by Fourier's integral-theorem (or, in a particular case, Fourier's series) just as radiation is analysed by the spectroscope.

Consider first a single one of these periodic constituents, say

$$A \sin \lambda x,$$

where A and λ are constants. We can without loss of generality suppose λ to be positive. The period of this term is $2\pi/\lambda$. We shall now show that *if this period is less than* 2w, *then an expression can be found which is cotabular with the given term and which has a period greater than* 2w.

For, *e.g.*, if the period lies between $2w$ and $2w/3$, so that λ lies between π/w and $3\pi/w$, the function

$$A \sin \left\{ \left(\lambda - \frac{2\pi}{w}\right)x + \frac{2\pi a}{w} \right\}$$

has the same values as $A \sin \lambda x$ when $x = a, a+w, a-w, a+2w$, etc.: and since λ lies between π/w and $3\pi/w$, we see that $(\lambda - 2\pi/w)$ lies between $-\pi/w$ and π/w, so the period of this new term is greater than $2w$. Similarly if the period of the given term lies between $2w/3$ and $2w/5$, so that λ lies between $3\pi/w$ and $5\pi/w$, then the function

$$A \sin \left\{ \left(\lambda - \frac{4\pi}{w}\right)x + \frac{4\pi a}{w} \right\}$$

is cotabular with $A \sin \lambda x$ and has a period greater than $2w$. Other possibilities can be treated in the same way, and the theorem stated is thus established.

We are thus led to the idea that if a function is given which can be analysed by Fourier's integral-theorem (or Fourier's series) into periodic constituents, then we can find another function which is cotabular with it and which has no constituents of period less than $2w$. That is to say, *we can replace the given function by a cotabular function in such a way as to remove all the rapid oscillations from it.*

§ 4. *Introduction of the cardinal function.*

We shall now carry out what has been indicated in the preceding article, namely, to analyse a given function into a number (generally an infinite number) of periodic constituents, then to replace the short-period components by long-period components which are cotabular with them, and finally to synthetise all the components into a new function. It will be shown later that this new function, which will be called the *cardinal function*, has certain remarkable properties.

Let $f(x)$ be the given function, from which all infinities except for imaginary infinite values of the argument are supposed to have been removed already by the method of § 2. Let $g(x, k)$ denote the function

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\mu\, f(\mu)\int_0^{\infty} e^{-\lambda k}\cos\lambda(x-\mu)d\lambda.$$

Here k denotes a positive constant, introduced for the purpose of securing convergence in the following developments.

Break up the range of integration in $g(x, k)$, thus—

$$g(x, k)=\frac{1}{\pi}\int_{-\infty}^{\infty} d\mu\, f(\mu)\left[\int_0^{\frac{\pi}{w}}+\int_{\frac{\pi}{w}}^{\frac{3\pi}{w}}+\int_{\frac{3\pi}{w}}^{\frac{5\pi}{w}}+\ .\ .\ .\ .\ \right]e^{-\lambda k}\cos\lambda(x-\mu)d\lambda.$$

The first partial integral consists of terms whose period in x is greater than $2w$, the second partial integral consists of terms whose periods are between $2w$ and $\frac{2}{3}w$, and so on. Replace every periodic term whose period is less than $2w$ by the corresponding cotabular term whose period is greater than $2w$, as explained in the preceding article. We thus obtain an expression which we shall denote by $G(x, k)$, where

$$G(x, k)=\frac{1}{\pi}\int_{-\infty}^{\infty} d\mu\, f(\mu)\left[\begin{array}{l}\displaystyle\int_0^{\frac{\pi}{w}} e^{-\lambda k}\cos\{\lambda(x-\mu)\}d\lambda \\ \displaystyle+\int_{-\frac{\pi}{w}}^{\frac{\pi}{w}} e^{-k\left(\lambda+\frac{2\pi}{w}\right)}\cos\left\{\lambda(x-\mu)+\frac{2\pi}{w}(a-\mu)\right\}d\lambda \\ \displaystyle+\int_{-\frac{\pi}{w}}^{\frac{\pi}{w}} e^{-k\left(\lambda+\frac{4\pi}{w}\right)}\cos\left\{\lambda(x-\mu)+\frac{4\pi}{w}(a-\mu)\right\}d\lambda \\ +\ .\ .\ .\ .\end{array}\right]$$

Summing the series of exponentials and cosines, we have

$$G(x, k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu\, f(\mu)\int_{-\frac{\pi}{w}}^{\frac{\pi}{w}} e^{-\lambda k}\frac{\sinh\frac{2\pi k}{w}\cos\left\{\lambda(x-\mu)\right\}-\sin\left\{\lambda(x-\mu)\right\}\sin\left\{\frac{2\pi}{w}(a-\mu)\right\}}{\cosh\frac{2\pi k}{w}-\cos\frac{2\pi}{w}(a-\mu)}d\lambda.$$

Performing the integration with respect to λ, this gives

$$G(x, k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu\, f(\mu)\left[\frac{\sin\frac{\pi}{w}(x-\mu-ik)\cos\frac{\pi}{w}(a-\mu-ik)}{(x-\mu-ik)\sin\frac{\pi}{w}(a-\mu-ik)}-\frac{\sin\frac{\pi}{w}(x-\mu+ik)\cos\frac{\pi}{w}(a-\mu+ik)}{(x-\mu+ik)\sin\frac{\pi}{w}(a-\mu+ik)}\right]$$

Now if we evaluate the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu\, f(\mu)\frac{e^{\frac{i\pi}{w}(x-\mu-ik)}}{x-\mu-ik}\ \frac{\cos\frac{\pi}{w}(a-\mu-ik)}{\sin\frac{\pi}{w}(a-\mu-ik)},$$

where k is positive, by Cauchy's Theorem of Residues, taking as contour the real axis of μ together with an infinite semicircle below the real axis, we obtain for it the value

$$\sum_{r=-\infty}^{\infty} \frac{f(a+rw-ik)e^{\frac{i\pi}{w}(x-a-rw)}}{\frac{\pi}{w}(x-a-rw)}.$$

Similarly if we evaluate the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu f(\mu)\frac{e^{-\frac{i\pi}{w}(x-\mu-ik)}}{x-\mu-ik}\frac{\cos\frac{\pi}{w}(a-\mu-ik)}{\sin\frac{\pi}{w}(a-\mu-ik)},$$

taking as contour the real axis of μ together with an infinite semicircle above the real axis, we obtain for it the value zero, since the integrand has no poles inside this contour.

Subtracting the latter result from the former, we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu f(\mu)\frac{\sin\frac{\pi}{w}(x-\mu-ik)}{x-\mu-ik}\frac{\cos\frac{\pi}{w}(a-\mu-ik)}{\sin\frac{\pi}{w}(a-\mu-ik)} = \frac{1}{2i}\sum_{r=-\infty}^{\infty}\frac{f(a+rw-ik).\,e^{\frac{i\pi}{w}(x-a-rw)}}{\frac{\pi}{w}(x-a-rw)}.$$

Similarly we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu f(\mu)\frac{\sin\frac{\pi}{w}(x-\mu+ik)}{x-\mu+ik}\frac{\cos\frac{\pi}{w}(a-\mu+ik)}{\sin\frac{\pi}{w}(a-\mu+ik)} = \frac{1}{2i}\sum_{r=-\infty}^{\infty}\frac{f(a+rw+ik).\,e^{-\frac{i\pi}{w}(x-a-rw)}}{\frac{\pi}{w}(x-a-rw)}$$

and thus we obtain

$$G(x, k) = \sum_{r=-\infty}^{\infty}\frac{f(a+rw-ik).\,e^{\frac{i\pi}{w}(x-a-rw)} - f(a+rw+ik).\,e^{-\frac{i\pi}{w}(x-a-rw)}}{2i\frac{\pi}{w}(x-a-rw)},$$

so that

$$\lim_{k\to 0} G(x, k) = \sum_{r=-\infty}^{\infty}\frac{f(a+rw)\sin\frac{\pi}{w}(x-a-rw)}{\frac{\pi}{w}(x-a-rw)}.$$

Now $G(x, k)$ is the function which was formed from $g(x, k)$ by replacing all the short-period terms by the corresponding cotabular long-period terms: and (as in Poisson's discussion of Fourier's integral) we have

$$f(x) = \lim_{k\to 0} g(x, k).$$

Hence we infer that *the expression*

$$\sum_{r=-\infty}^{-\infty}\frac{f(a+rw)\sin\frac{\pi}{w}(x-a-rw)}{\frac{\pi}{w}(x-a-rw)} \quad . \quad . \quad . \quad . \quad (3)$$

or

$$\frac{w}{\pi}\sin\frac{\pi}{w}(x-a)\sum_{r=-\infty}^{\infty}\frac{(-1)^r f(a+rw)}{x-a-rw}. \quad . \quad . \quad . \quad . \qquad (4)$$

represents a function which is cotabular with the given function f(x), *but which has no periodic constituents of period less than* 2w.

Now, in order to construct the expression (3) or (4), we do not need to know anything about $f(x)$ except its values $f(a), f(a+w), f(a-w)$, etc., at the tabulated values of the argument. These values, however, are not peculiar to $f(x)$, but are common to the whole set of cotabular functions. It follows that *we arrive at the same expression* (3) *whatever function* f(x) *of the cotabular set we start from.* The expression (3) is therefore an *invariantive function* of the cotabular set: and it may be regarded as the *simplest function* belonging to the set. We shall call it the CARDINAL FUNCTION of the set.

§ 5. *Examples of the determination of a cardinal function.*

We shall now work out two examples in order to show how in any given case the cardinal function may be obtained from the formula (4).

Example 1.—Suppose that the given tabular values of the function $f(x)$ are as follows:—

$$f(0)=0, \quad f(1)=-1, \quad f(2)=\tfrac{1}{2}, \qquad f(3)=-\tfrac{1}{3}, \ . \ . \ . \ . \ f(n)=\frac{(-1)^n}{n}, \ . \ . \ . \ .$$

$$f(-1)=1, \quad f(-2)=-\tfrac{1}{2}, \quad f(-3)=\tfrac{1}{3}, \ . \ . \ . \ . \ f(-n)=\frac{(-1)^{n+1}}{n}, \ . \ . \ . \ .$$

The corresponding cardinal function is, by formula (4),

$$\frac{1}{\pi}\sin\pi x\left[\begin{array}{l}\dfrac{1}{x-1}+\dfrac{1}{2(x-2)}+\dfrac{1}{3(x-3)}+\dfrac{1}{4(x-4)}+\ . \ . \ . \ . \\ -\dfrac{1}{x+1}-\dfrac{1}{2(x+2)}-\dfrac{1}{3(x+3)}-\dfrac{1}{4(x+4)}+\ . \ . \ . \ .\end{array}\right]$$

or, summing the series,

$$\frac{\sin\pi x}{\pi x}\left[\frac{\Gamma'(1-x)}{\Gamma(1-x)}-\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right]$$

or

$$-\frac{\sin\pi x}{\pi x}\frac{d}{dx}\log\{\Gamma(1+x)\Gamma(1-x)\}$$

or

$$\frac{\sin\pi x}{\pi x}\frac{d}{dx}\log\frac{\sin\pi x}{\pi x}$$

or

$$\frac{\cos\pi x}{x}-\frac{\sin\pi x}{\pi x^2}.$$

This is the required cardinal function. It is the only analytic function having the above tabular values which has no singularities in the finite part of the x-plane and no oscillations of period less than 2.

Example 2.—Suppose that the given tabular values of the function $f(x)$ are as follows:—

$$f(a)=0,\ f(a+w)=1,\quad f(a+2w)=1,\quad f(a+3w)=0,\ f(a+4w)=-1,\ \dots\dots$$
$$f(a-w)=-1,\ f(a-2w)=-1,\ f(a-3w)=0,\ f(a-4w)=1,\ \dots\dots,$$

so that by (4) the cardinal function is in this case

$$\frac{w}{\pi}\sin\left\{\frac{\pi}{w}(x-a)\right\}\left[\frac{-1}{x-a+w}+\frac{1}{x-a+w}+\frac{1}{x-a-2w}-\frac{1}{x-a+2w}-\frac{1}{x-a-4w}\right.$$
$$\left.+\frac{1}{x-a+4w}+\frac{1}{x-a-5w}-\dots\dots\right]$$

Now remembering that

$$\cot\frac{\pi(x-a-w)}{3w}=\frac{3w}{\pi}\left\{\frac{1}{x-a-w}+\frac{1}{x-a+2w}+\frac{1}{x-a-4w}+\frac{1}{x-a+5w}\right.$$
$$\left.+\frac{1}{x-a-7w}+\dots\dots\right\}$$

and

$$\cot\frac{\pi(x-a+w)}{3w}=\frac{3w}{\pi}\left\{\frac{1}{x-a+w}+\frac{1}{x-a+4w}+\frac{1}{x-a-2w}+\frac{1}{x-a+7w}\right.$$
$$\left.+\frac{1}{x-a-5w}+\dots\dots\right\}$$

we see that this cardinal function is *

$$\frac{1}{3}\sin\left\{\frac{\pi}{w}(x-a)\right\}\left[\cot\frac{\pi(x-a+w)}{3w}-\cot\frac{\pi(x-a-w)}{3w}\right]$$

or

$$\frac{-\frac{1}{3}\sin\frac{\pi(x-a)}{w}\sin\frac{2\pi}{3}}{\sin\frac{\pi(x-a+w)}{3w}\sin\frac{\pi(x-a-w)}{3w}};$$

so, making use of the identity

$$\sin 3\chi=-4\sin\chi\sin\left(\chi+\frac{\pi}{3}\right)\sin\left(\chi-\frac{\pi}{3}\right),$$

we obtain the cardinal function corresponding to the above tabular values in the simple form

$$\frac{2}{\sqrt{3}}\sin\frac{\pi(x-a)}{3w}.$$

It will be noticed that in Example 1 the tabular values of the function tend to the limit zero for infinite values of the argument, whereas in Example 2 they do not tend to the limit zero.

* It is not in general permissible to alter the order of the terms in a conditionally convergent series: but it may readily be proved that in the present case the value of the sum is not altered by the particular rearrangement which is made.

§ 6. *Direct proof of the properties of the cardinal function.*

Let C(x) denote the cardinal function associated with a given function $f(x)$, so that

$$C(x)=\sum_{r=-\infty}^{\infty}\frac{f(a+rw)\sin\left\{\frac{\pi}{w}(x-a-rw)\right\}}{\frac{\pi}{w}(x-a-rw)}.$$

Then we can prove the characteristic properties of this function directly.

1°. C(x) *is cotabular with* f(x).

For the expression $\dfrac{\sin\left\{\frac{\pi}{w}(x-a-rw)\right\}}{\frac{\pi}{w}(x-a-rw)}$ has the value unity when $x\to(a+rw)$, and has the value zero when x has any other one of the values a, $a+w$, $a-w$, $a+2w$, . . . From this it follows at once that

$$C(a+rw)=f(a+rw)\qquad (r=0,\ \pm1,\ \pm2,\ \pm3,\ \ldots\ .),$$

which establishes the property of cotabularity.

2°. C(x) *has no singularities in the finite part of the* x-*plane.*

For a singularity at any point would give rise to a failure of convergence of the series (3) at that point: but its convergence, for the class of functions $f(x)$ considered, can readily be deduced from its mode of origin as a sum of residues.

3°. *When* C(x) *is analysed into periodic constituents by Fourier's integral-theorem, all constituents of period less than* 2w *are absent.*

For if we resolve the function

$$\frac{\sin\left\{\frac{\pi}{w}(x-c)\right\}}{\frac{\pi}{w}(x-c)}\qquad\qquad (5)$$

(where c denotes any constant) into periodic constituents by Fourier's integral-theorem, we have

$$\frac{\sin\left\{\frac{\pi}{w}(x-c)\right\}}{\frac{\pi}{w}(x-c)}=\frac{1}{\pi}\int_0^{\infty}d\lambda\int_{-\infty}^{\infty}d\mu\,\frac{\sin\left\{\frac{\pi}{w}(\mu-c)\right\}\cos\left\{\lambda(x-\mu)\right\}}{\frac{\pi}{w}(\mu-c)}$$

$\left(\text{writing } y \text{ for } \dfrac{\pi(\mu-c)}{w}\right)$

$$=\frac{w}{\pi^2}\int_0^{\infty}d\lambda\int_{-\infty}^{\infty}dy\,\frac{\sin y\cos\left(\lambda x-\lambda c-\frac{\lambda w}{\pi}y\right)}{y}.$$

Now it is well known that

$$\int_{-\infty}^{\infty} \frac{\sin y \cos ky}{y} dy$$

is zero when $k > 1$: and

$$\int_{-\infty}^{\infty} \frac{\sin y \sin ky}{y} dy$$

is always zero. Hence in the above repeated integral the first integration gives a zero result so long as $\lambda w > \pi$; that is to say, there are in the expression (5) no constituents of the type $\cos\{\lambda(x-\mu)\}$ for which $\lambda w > \pi$, and for which therefore the period is less than $2w$.

The theorem being thus seen to be true for every single term of the series (3), is consequently true for the cardinal function as a whole.

We may remark in passing that it is possible to construct an infinite number of functions cotabular with $f(x)$ by means of series more or less resembling the series (3): for instance, the function

$$\sum_{r=-\infty}^{\infty} \left[e^{-c(x-a-rw)^{2m}} f(a+rw) \left\{ \frac{\sin \frac{\pi}{w}(x-a-rw)}{\frac{\pi}{w}(x-a-rw)} \right\}^{n} \right]$$

where c denotes any real positive constant, and m and n denote any positive integers, is a function cotabular with $f(x)$. But this function does not possess the property characteristic of the cardinal function, namely, that periodic constituents of period less than $2w$ are absent. Such functions are, however, all of them solutions of the problem "*To find an analytical expression for a function when we know the values which it has for the values* a, a+w, a−w, a+2w . . . *of its argument*": which is essentially the fundamental problem of the theory of interpolation.

§ 7. *Solution of the questions proposed in* § 1.

We are now in a position to answer the first of the questions proposed in § 1, as to *which of the functions of the cotabular set is represented by the expansion*

$$f_0 + n\,\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2!}\delta^2 f_0 + \frac{(n+1)n(n-1)}{3!}\delta^3 f_{\frac{1}{2}} + \frac{(n+1)n(n-1)(n-2)}{4!}\delta^4 f_0 + \ldots .$$

The answer is that *this expansion represents the cardinal function.* This we shall now prove.

Consider the algebraical identity

$$\frac{1}{z-a-nw}=\frac{1}{z-a}+\frac{nw}{(z-a)(z-a-w)}+\frac{n(n-1)w^2}{(z-a+w)(z-a)(z-a-w)}$$
$$+\frac{(n+1)n(n-1)w^3}{(z-a+w)(z-a)(z-a-w)(z-a-2w)}+\ldots\ldots$$
$$+\frac{n(n^2-1^2)(n^2-2^2)\ldots\ldots\{n^2-(r-1)^2\}(n-r)w^{2r}}{(z-a)\{(z-a)^2-w^2\}\{(z-a)^2-2^2w^2\}\ldots\ldots\{(z-a)^2-r^2w^2\}}$$
$$+\frac{n(n^2-1^2)(n^2-2^2)\ldots\ldots(n^2-r^2)w^{2r+1}}{(z-a)\{(z-a)^2-w^2\}\ldots\ldots\{(z-a)^2-r^2w^2\}(z-a-nw)} \quad (6)$$

Let $f(x)$ be the given function, and let $C(x)$ be the corresponding cardinal function. Multiply the identity (6) throughout by $\frac{1}{2\pi i}C(z)$, and integrate with respect to z round any simple contour γ which encloses all the points a, $a+w$, $a-w$, $a+2w$, ..., $a+rw$, $a-rw$, $a+nw$.

Now we have

$$\frac{1}{2\pi i}\int_\gamma \frac{C(z)dz}{z-a-nw}=C(a+nw)$$
$$\frac{1}{2\pi i}\int_\gamma \frac{C(z)dz}{z-a}=C(a)=f_0$$
$$\frac{w}{2\pi i}\int_\gamma \frac{C(z)dz}{(z-a)(z-a-w)}=C(a+w)-C(a)=f_1-f_0=\delta f_{\frac{1}{2}}$$
$$\frac{2!\,w^2}{2\pi i}\int_\gamma \frac{C(z)dz}{(z-a+w)(z-a)(z-a-w)}=\delta^2 f_0, \text{ etc.}$$

Thus the equation (6) becomes

$$C(a+nw)=f_0+n\delta f_{\frac{1}{2}}+\frac{n(n-1)}{2!}\delta^2 f_0+\frac{(n+1)n(n-1)}{3!}\delta^3 f_{\frac{1}{2}}+\ldots\ldots$$
$$+\frac{n(n^2-1^2)(n^2-2^2)\ldots\ldots\{n^2-(r-1)^2\}(n-r)}{2r!}\delta^{2r}f_0$$
$$+\frac{1}{2\pi i}\int_\gamma \frac{n(n^2-1^2)(n^2-2^2)\ldots\ldots(n^2-r^2)w^{2r+1}C(z)dz}{(z-a)\{(z-a)^2-w^2\}\ldots\ldots\{(z-a)^2-r^2w^2\}(z-a-nw)} \quad (7)$$

We have now to investigate the value of the last term in the right-hand side as r increases indefinitely. Since $C(z)$ has no singularity in the finite part of the plane, we are free to extend the contour as much as we like. We can suppose it to be a circle of very large radius, whose centre is at $a+nw$.

Now the integrand, apart from the factor $\frac{C(z)}{z-a-nw}$, may be written

$$\frac{n\left(1-\frac{n^2}{1^2}\right)\left(1-\frac{n^2}{2^2}\right)\ldots\ldots\left(1-\frac{n^2}{r^2}\right)}{\left(\frac{z-a}{w}\right)\left\{1-\frac{(z-a)^2}{1^2w^2}\right\}\left\{1-\frac{(z-a)^2}{2^2w^2}\right\}\ldots\ldots\left\{1-\frac{(z-a)^2}{r^2w^2}\right\}},$$

and when r increases indefinitely this tends to the value

$$\frac{\sin \pi n}{\sin \dfrac{\pi(z-a)}{w}},$$

so that the integral to be studied is essentially

$$\sin \pi n \int_\gamma \frac{C(z)dz}{(z-a-nw)\sin \dfrac{\pi(z-a)}{w}} \quad . \quad . \quad . \quad . \quad (8)$$

The question as to whether this integral tends to zero or not depends fundamentally on whether $C(z)$ becomes infinite to a lower or higher order than $\sin \frac{\pi(z-a)}{w}$, when the imaginary part of z tends to infinity. Now a simple periodic function like $\sin \lambda z$ becomes infinite to the same order as $e^{\lambda y}$, where y denotes the modulus of the imaginary part of z: and we have seen that the distinguishing property of the cardinal function $C(z)$ is that the periodic constituents into which it can be analysed all have periods greater than $2w$: so, combining these statements, we see that $C(z)$ becomes infinite to an order less than $e^{\frac{\pi}{w}y}$, whereas $\sin \frac{\pi(z-a)}{w}$ becomes infinite to the order $e^{\frac{\pi}{w}y}$. Thus the factor

$$\frac{C(z)}{\sin \dfrac{\pi(z-a)}{w}}$$

of the integrand tends to zero when the imaginary part of z tends to either positive or negative infinity: and as the other factor $\frac{dz}{z-a-nw}$ may be written $id\theta$, where θ denotes the vectional angle of the point z measured from the origin $a+nw$, we see by a proof of the kind usual in analysis that *the integral* (8) *vanishes.** The equation (7) now becomes

$$C(a+nw) = f_0 + n\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2\,!}\delta^2 f_0 + \frac{(n+1)n(n-1)}{3\,!}\delta^3 f_{\frac{1}{2}} + \frac{(n+1)n(n-1)(n-2)}{4\,!}\delta^4 f_0 + \ldots \; ad.\ inf.,$$

which shows that *the function represented by the expansion is the cardinal function of the cotabular set which is associated with the given function* f(x).†

* The manner in which the characteristic properties of the cardinal-function are required in order to ensure the vanishing of this remainder-term is very remarkable.

† It should be noted that the interpolation-expansion considered is a "central-difference" formula, *i.e.* it makes use of *all* the tabulated values of $f(x)$ both above and

The first question proposed in § 1 is thus answered; and the answer to the second question follows from it, since we have seen in §§ 4–5 how the cardinal function may be constructed analytically.

§ 8. *Conclusion.*

The cardinal function may be regarded from many different points of view. We defined it originally as that (unique) function of the cotabular set which has no singularities in the finite part of the plane and no constituents whose period is less than twice the tabular interval w. But the result of § 7 shows that it might be defined as the sum of the central-difference expansion formed with the given set of tabular values: or (what amounts ultimately to the same thing) it might be defined as the limit, when $r \to \infty$, of that polynomial in x of degree $2r$ which has the values $f_0, f_1, f_{-1}, f_2, f_{-2}, \ldots, f_r, f_{-r}$ when the argument has the values $a, a+w, a-w, \ldots, a+rw, a-rw$ respectively. When we regard it from this latter point of view, we see the underlying reason for the absence of singularities in the finite part of the plane and of short-period oscillations.

The introduction of the cardinal function seems to necessitate some reconstruction of ideas in the general theory of the representation of an arbitrary function by a series of given polynomials, say

$$f(x) = a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) + \ldots \text{ ad inf.}$$

Our ideas on the subject of these expansions have hitherto been based chiefly on the study of the two best-known cases, namely, Taylor's expansion

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \ldots,$$

and the expansion in terms of Legendre functions

$$f(x) = a_0 \mathrm{P}_0(x) + a_1 \mathrm{P}_1(x) + a_2 \mathrm{P}_2(x) + \ldots$$

Now it so happens that in both these special cases the roots of the given polynomials are either all concentrated in a single point (as in Taylor's expansion) or else everywhere-dense on a finite segment of the real axis (as in the Legendre case, the roots of $\mathrm{P}_n(x)$ when $n \to \infty$ being everywhere-

below a. In the case of an interpolation-formula such as Newton's, namely,

$$f(a+nw) = f_0 + n\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2!}\delta^2 f_1 + \frac{n(n-1)(n-2)}{3!}\delta^3 f_{\frac{3}{2}} + \ldots,$$

use is made only of the tabulated values of $f(x)$ for the values $a, a+w, a+2w, \ldots$ of the argument, and no use is made of the tabulated values of $f(x)$ for the values $a-w, a-2w, a-3w, \ldots$ of the argument; in such cases a wholly different theorem holds, which I hope to give in a later paper.

dense on the segment of the real axis between -1 and $+1$). In such cases the coefficients $a_0, a_1, a_2, \ldots$ of the expansion can be determined in terms of $f(x)$, and there is no doubt as to what function is represented by the expansion so long as it converges—there is nothing analogous to the property of cotabularity. When, however, the roots of the polynomials, instead of being everywhere-dense on a segment, are distributed discretely over the whole infinite length of the real axis of x (as is the case in the expansion

$$f_0 + n\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2!}\delta^2 f_0 + \frac{(n+1)n(n-1)}{3!}\delta^3 f_{\frac{1}{2}} + \ldots,$$

where the polynomials are 1, n, $n(n-1)$, $(n+1)n(n-1)$, etc.), it seems probable that a property analogous to cotabularity will come into evidence, and the theory of the expansion will depend essentially on a "cardinal function" analogous to that introduced above.

The results of the present paper suggest another development. For long past the applied mathematicians have complained that Pure Mathematics is daily becoming more complicated and harder to understand. This complaint refers chiefly to the increased rigour with which the theories of Analysis are now expounded, and which is closely connected with the extension of knowledge regarding discontinuities, singularities, and other phenomena of which the older mathematics took no account. Indeed, the modern Theory of Functions of a Real Variable is concerned largely with cases in which the distribution of fluctuations and singularities transcends all intuitive or geometrical representation. It seems possible that some of the difficulties of such cases might be avoided by the introduction of a function analogous to the "cardinal function" of the present paper, which would be simpler than the function under discussion, but would be equal to it for an infinite number of values of the variable, and could be substituted for it in all practical and some theoretical investigations.

(*Issued separately July* 13, 1915.)