Philosophical Magazine Series 5
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm16

XXXIII. On the electric and magnetic effects produced by the motion of electrified bodies

J.J. Thomson M.A. [a]
[a] Trinity College, Cambridge
Published online: 08 Jun 2010.

To cite this article: J.J. Thomson M.A. (1881) XXXIII. On the electric and magnetic effects produced by the motion of electrified bodies , Philosophical Magazine Series 5, 11:68, 229-249, DOI: 10.1080/14786448108627008

To link to this article: http://dx.doi.org/10.1080/14786448108627008

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

APRIL 1881.

XXXIII. *On the Electric and Magnetic Effects produced by the Motion of Electrified Bodies.* *By* J. J. THOMSON, *B.A., Fellow of Trinity College, Cambridge**.

§ 1. IN the interesting experiments recently made by Mr. Crookes (Phil. Trans. 1879, parts 1 and 2) and Dr. Goldstein (Phil. Mag. Sept. and Oct. 1880) on "Electric Discharges in High Vacua," particles of matter highly charged with electricity and moving with great velocities form a prominent feature in the phenomena; and a large portion of the investigations consists of experiments on the action of such particles on each other, and their behaviour when under the influence of a magnet. It seems therefore to be of some interest, both as a test of the theory and as a guide to future experiments, to take some theory of electrical action and find what, according to it, is the force existing between two moving electrified bodies, what is the magnetic force produced by such a moving body, and in what way the body is affected by a magnet. The following paper is an attempt to solve these problems, taking as the basis Maxwell's theory that variations in the electric displacement in a dielectric produce effects analogous to those produced by ordinary currents flowing through conductors.

For simplicity of calculation we shall suppose all the moving bodies to be spherical.

* Communicated by the Author.

§ 2. The first case we shall consider is that of a charged sphere moving through an unlimited space filled with a medium of specific inductive capacity K.

The charged sphere will produce an electric displacement throughout the field; and as the sphere moves the magnitude of this displacement at any point will vary. Now, according to Maxwell's theory, a variation in the electric displacement produces the same effect as an electric current; and a field in which electric currents exist is a seat of energy; hence the motion of the charged sphere has developed energy, and consequently the charged sphere must experience a resistance as it moves through the dielectric. But as the theory of the variation of the electric displacement does not take into account any thing corresponding to resistance in conductors, there can be no dissipation of energy through the medium; hence the resistance cannot be analogous to an ordinary frictional resistance, but must correspond to the resistance theoretically experienced by a solid in moving through a perfect fluid. In other words, it must be equivalent to an increase in the mass of the charged moving sphere, which we now proceed to calculate.

Let a be the radius of the moving sphere, e the charge on the sphere, and let us suppose that the sphere is moving parallel to the axis of x with the velocity p; let ξ, η, ζ be the coordinates of the centre of the sphere; let f, g, h be the components of the electric displacement along the axes of x, y, z respectively at a point whose distance from the centre of the sphere is ρ, ρ being greater than a. Then, neglecting the self-induction of the system (since the electromotive forces it produces are small compared with those due to the direct action of the charged sphere), we have

$$f = -\frac{e}{4\pi}\frac{d}{dx}\frac{1}{\rho},$$

$$g = -\frac{e}{4\pi}\frac{d}{dy}\frac{1}{\rho},$$

$$h = -\frac{e}{4\pi}\frac{d}{dz}\frac{1}{\rho};$$

therefore

$$\frac{df}{dt} = -\frac{ep}{4\pi}\frac{d^2}{dx\,d\xi}\frac{1}{\rho},$$

$$\frac{dg}{dt} = -\frac{ep}{4\pi}\frac{d^2}{d\xi\,dy}\frac{1}{\rho},$$

$$\frac{dh}{dt} = -\frac{e}{4\pi}\frac{d^2}{d\xi\,dz}\frac{1}{\rho};$$

hence

$$\left.\begin{aligned}\frac{df}{dt}&=\frac{ep}{4\pi}\frac{d^2}{dx^2}\frac{1}{\rho},\\ \frac{dg}{dt}&=\frac{ep}{4\pi}\frac{d^2}{dx\,dy}\frac{1}{\rho},\\ \frac{dh}{dt}&=\frac{ep}{4\pi}\frac{d^2}{dx\,dz}\frac{1}{\rho}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Using Maxwell's notation, let F, G, H be the components of the vector-potential at any point; then, by 'Electricity and Magnetism,' § 616,

$$F=\mu\iiint\frac{u}{\rho'}\,dx\,dy\,dz,$$

$$G=\mu\iiint\frac{v}{\rho'}\,dx\,dy\,dz,$$

$$H=\mu\iiint\frac{w}{\rho'}\,dx\,dy\,dz,$$

where u, v, w are the components of the electric current through the element $dx\,dy\,dz$, and ρ' is the distance of that element from the point at which the values of F, G, H are required, μ is the coefficient of magnetic permeability. In the case under consideration,

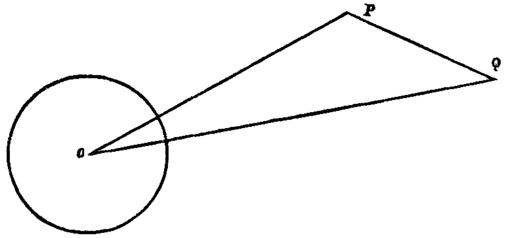
$$F=\mu\iiint\frac{\frac{df}{dt}}{\rho'}\,dx\,dy\,dz\,;$$

substituting for $\frac{df}{dt}$ its value from equation (1), we get

$$F=\frac{\mu ep}{4\pi}\iiint\frac{1}{\rho'}\frac{d^2}{dx^2}\frac{1}{\rho}\,dx\,dy\,dz,$$

with similar expressions for G and H.

Let us proceed to calculate the value of F at a point P.



Let O be the centre of the sphere; then $OQ=\rho$, $PQ=\rho'$, $OP=R$,

$$F=\frac{\mu ep}{4\pi}\iiint\frac{1}{PQ}\frac{d^2}{dx^2}\frac{1}{\rho}\,dx\,dy\,dz.$$

Now $\frac{d^2}{dx^2}\frac{1}{\rho}=\frac{Y_2}{\rho^3}$, where Y_2 is a surface harmonic of the second order. And when $\rho > R$,

$$\frac{1}{PQ}=\frac{1}{\rho}+\frac{R}{\rho^2}Q_1+\frac{R^2}{\rho^3}Q_2+\dots;$$

and when $\rho < R$,

$$\frac{1}{PQ}=\frac{1}{R}+\frac{\rho}{R^2}Q_1+\frac{\rho^2}{R^3}Q_2+\dots;$$

where Q_1, Q_2, &c. are zonal harmonics of the first and second orders respectively referred to OP as axis.

Let Y'_2 denote the value of Y_2 along OP. Then, since $\int Y_n Q_m\,ds$, integrated over a sphere of unit radius, is zero when n and m are different, and $\frac{4\pi}{2n+1}Y'_n$ when $n=m$, Y'_n being the value of Y_n at the pole of Q_n, and since there is no electric displacement within the sphere,

$$F=\frac{\mu ep}{4\pi}\times\frac{4\pi Y'_2}{5}\left\{\int_R^\infty\frac{R^2}{\rho^4}d\rho+\int_a^R\frac{\rho d\rho}{R^3}\right\}$$

$$=\frac{\mu ep}{5}Y'_2\left(\frac{5}{6R}-\frac{a^2}{2R^3}\right),$$

or, as it is more convenient to write it,

$$=\frac{\mu ep}{5}\left(\frac{5R^2}{6}-\frac{a^2}{2}\right)\frac{d^2}{dx^2}\frac{1}{R}.$$

By symmetry, the corresponding values of G and H are

$$G=\frac{\mu ep}{5}\left(\frac{5R^2}{6}-\frac{a^2}{2}\right)\frac{d^2}{dx\,dy}\frac{1}{R},$$

$$H=\frac{\mu ep}{5}\left(\frac{5R^2}{6}-\frac{a^2}{2}\right)\frac{d^2}{dx\,dz}\frac{1}{R}.$$

These values, however, do not satisfy the condition

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0.$$

If, however, we add to F the term $\frac{2\mu ep}{3R}$, this condition will be satisfied; while, since the term satisfies Laplace's equation, the other conditions will not be affected: thus we have finally for points outside the sphere,

$$\left.\begin{aligned} F &= \frac{\mu ep}{5}\left(\frac{5R^2}{6} - \frac{a^2}{2}\right)\frac{d^2}{dx^2}\frac{1}{R} + \frac{2\mu ep}{3R}, \\ G &= \frac{\mu ep}{5}\left(\frac{5R^2}{6} - \frac{a^2}{2}\right)\frac{d^2}{dx\,dy}\frac{1}{R}, \\ H &= \frac{\mu ep}{5}\left(\frac{5R^2}{6} - \frac{a^2}{2}\right)\frac{d^2}{dx\,dz}\frac{1}{R}. \end{aligned}\right\} \quad . \quad . \quad (2)$$

Now, by 'Electricity and Magnetism,' § 634, T the kinetic energy

$$= \tfrac{1}{2}\iiint(Fu + Gv + Hw)dx\,dy\,dz,$$

in our case,

$$= \tfrac{1}{2}\iiint\left(F\frac{df}{dt} + G\frac{dg}{dt} + H\frac{dh}{dt}\right)dx\,dy\,dz.$$

Now

$$\tfrac{1}{2}\iiint F\frac{df}{dt}dx\,dy\,dz,$$

substituting for F and $\frac{df}{dt}$,

$$= \tfrac{1}{2}\frac{\mu e^2p^2}{20\pi}\iiint\left(\frac{5r^2}{6} - \frac{a^2}{2}\right)\left(\frac{d^2}{dx^2}\frac{1}{r}\right)^2 dx\,dy\,dz,$$

since the term

$$\frac{\mu e^2p^2}{12\,.\,\pi}\iiint\frac{1}{r}\frac{d^2}{dx^2}\frac{1}{r}\,dx\,dy\,dz$$

evidently vanishes.

Transforming to polars and taking the axis of x as the initial line, the above integral

$$= \frac{\mu e^2p^2}{40\pi}\int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\left(\frac{5r^2}{6} - \frac{a^2}{2}\right)\frac{(3\cos^2\theta - 1)^2}{r^4}\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{4\mu e^2p^2}{75a},$$

$$\tfrac{1}{2}\iiint G\frac{dg}{dt}dx\,dy\,dz = \frac{\mu e^2p^2}{40\pi}\iiint\left(\frac{5}{6}r^2 - \frac{a^2}{2}\right)\left(\frac{d^2}{dx\,dy}\frac{1}{r}\right)^2 dx\,dy\,dz.$$

By transforming to polars, as before, we may show that this

$$= \frac{\mu e^2p^2}{25a}.$$

Similarly,

$$\tfrac{1}{2}\iiint H\frac{dh}{dt}dx\,dy\,dz = \frac{\mu e^2p^2}{25a};$$

$\therefore$ T, the kinetic energy due to the electrification

$$=\tfrac{1}{2}\iiint\left(\mathrm{F}\frac{df}{dt}+\mathrm{G}\frac{dg}{dt}+\mathrm{H}\frac{dh}{dt}\right)dx\,dy\,dz$$

$$=\frac{2\mu e^2p^2}{15a}.$$

Hence, if m be the mass of the sphere, the whole kinetic energy

$$=\left(\frac{m}{2}+\frac{2}{15}\frac{\mu e^2}{a}\right)p^2; \quad \ldots\ldots \quad (3)$$

or the effect of the electrification is the same as if the mass of the sphere were increased by $\frac{4}{15}\frac{\mu e^2}{a}$, or, if V be the potential of the sphere, by $\frac{4}{15}\mu\mathrm{K}^2\mathrm{V}^2a$.

To form some idea of what the increase of mass could amount to in the most favourable case, let us suppose the earth electrified to the highest potential possible without discharge, and calculate the consequent increase in mass. According to Dr. Macfarlane's experiments, published in the Philosophical Magazine for December 1880, the electric force in air at ordinary temperatures and pressures must not exceed 3×10^{12} (electromagnetic system of units). The electric force just outside the sphere is V/a; hence the greatest possible value of V is $3\times10^{12}a$, where a is the radius of the earth. Putting this value for V, $\mu=1$, $\mathrm{K}=\frac{1}{9\cdot10^{20}}$, $a=6\cdot4\times10^8$, we get for the corresponding value of the increase of mass 7×10^8 grms., or about 650 tons, a mass which is quite insignificant when compared with the mass of the earth.

For spheres of different sizes, the greatest increase in mass varies as the cube of the radius; hence the ratio of this increase to the whole mass of the sphere is constant for all spheres of the same material; for spheres of different materials the ratio varies inversely as the density of the material.

If the body moves so that its velocities parallel to the axes of x, y, z respectively are p, q, r, then it is evident that the effect of the electrification will be equivalent to an increase of $\frac{4}{15}\mu\mathrm{K}^2\mathrm{V}^2a(p^2+q^2+r^3)$ in the mass of the sphere.

§ 3. To find the magnetic force produced by the moving sphere at any point in the field. By equations (2) we have, for points outside the sphere,

$$F=\frac{\mu ep}{5}\left(\frac{5R^2}{6}-\frac{a^2}{2}\right)\frac{d^2}{dx^2}\frac{1}{R}+\frac{\mu ep}{3}\frac{2}{R},$$

$$G=\frac{\mu ep}{5}\left(\frac{5R^2}{6}-\frac{a^2}{2}\right)\frac{d^2}{dx\,dy}\frac{1}{R},$$

$$H=\frac{\mu ep}{5}\left(\frac{5R^2}{6}-\frac{a^2}{2}\right)\frac{d^2}{dx\,dr}\frac{1}{R}.$$

Now if α, β, γ be the components of the magnetic induction at the point (x, y, z),

$$\alpha=\frac{dH}{dy}-\frac{dG}{dz}=0,$$

$$\beta=\frac{dF}{dz}-\frac{dH}{dx}=-\frac{\mu epz}{R^3}=\mu ep\frac{d}{dz}\frac{1}{R},$$

$$\gamma=\frac{dG}{dx}-\frac{dF}{dy}=\frac{\mu epy}{R^3}=-\mu ep\frac{d}{dy}\frac{1}{R}.$$

Hence we see, by symmetry, that if the sphere move with velocity q parallel to the axis of y, the corresponding values would be

$$\alpha=-\mu eq\frac{d}{dz}\frac{1}{R},$$

$$\beta=0,$$

$$\gamma=\mu eq\frac{d}{dx}\frac{1}{R};$$

and if it moved with velocity r parallel to the axis of z, the corresponding values would be

$$\alpha=\mu er\frac{d}{dy}\frac{1}{R},$$

$$\beta=-\mu er\frac{d}{dx}\frac{1}{R},$$

$$\gamma=0.$$

Hence, if p, q, r be the components of the velocity of the centre of the sphere parallel to the axes of x, y, z respectively, the components of magnetic induction are

$$\alpha=\mu e\left(r\frac{d}{dy}\frac{1}{R}-q\frac{d}{dz}\frac{1}{R}\right),$$

$$\beta=\mu e\left(p\frac{d}{dz}\frac{1}{R}-r\frac{d}{dx}\frac{1}{R}\right),$$

$$\gamma=\mu e\left(q\frac{d}{dx}\frac{1}{R}-p\frac{d}{dy}\frac{1}{R}\right);$$

or they may also be written

$$\left.\begin{aligned}\alpha&=\frac{\mu e}{R^3}\big(q(z-\zeta)-r(y-\eta)\big),\\ \beta&=\frac{\mu e}{R^3}\big(r(x-\xi)-p(z-\zeta)\big),\\ \gamma&=\frac{\mu e}{R^3}\big(p(y-\eta)-q(x-\xi)\big).\end{aligned}\right\} \quad . \quad . \quad (4)$$

Comparing these expressions with those given by Ampère for the magnetic force produced by a current, we see that the magnetic force due to the moving sphere is the same as that produced per unit length of a current whose intensity is $\mu e\sqrt{p^2+q^2+r^2}$, situated at the centre of the sphere, the direction of the positive current coinciding with the direction of motion of the sphere. The resultant magnetic force produced by the sphere at any point is $\omega\mu e \sin\epsilon/\rho^2$, ω being the velocity of the sphere, and ϵ the angle between the direction of motion of the sphere and the radius vector ρ drawn from the centre of the sphere to the point; the direction of the force is perpendicular both to the direction of motion of the sphere and the radius vector from the centre of the sphere to the point; and the direction of the force and the direction of motion are related to each other like translation and rotation in a right-handed screw.

It may be useful to form a rough numerical idea of the magnitude of the greatest magnetic force which could be produced by a moving charged sphere. The greatest value of the force $=\mu KVa\omega/\rho^2$, where a is the radius and V the potential of the sphere. Now if F be the greatest electric force which can exist without discharge, the greatest value of V is Fa. According to Mr. Macfarlane's experiments F is, roughly speaking, about 3×10^{12}, $\mu K=\frac{1}{9\times10^{20}}$; substituting these values, the greatest value of the magnetic force becomes $\frac{1a^2\omega}{3\rho^2 10^8}$. Now $\frac{a}{\rho}$ cannot be greater than unity; so the greatest value of the force is $\omega/3\times10^8$. If the sphere were attached to an arm of such length that it described a metre in each complete revolution of the arm, and if the arm were to make 100 revolutions a second, ω would equal 10^4, and the greatest magnetic force would be $1/3\times10^4=\cdot000033$. Prof. Rowland, in his experiments on the magnetic effects of electric convection, measured a magnetic force only about one tenth of this.

The result we have just obtained (viz. that a moving body charged with electricity produces the effect of an electric current) shows that Prof. Rowland's experiments on electric convection are in agreement with Maxwell's theory.

§ 4. The fact that a moving body charged with electricity produces a vector-potential in the field through which it is moving, suggests a possible theory of the cause of the green phosphorescence observed in vacuum-tubes at the places where the molecular streams strike the glass, different from that put forward by Mr. Crookes. It will be seen from the above work that the moving particle produces a vector-potential whose value depends on the velocity of the moving body. Now, when a particle strikes the glass directly, its velocity is reversed and the vector-potential changes sign; thus during the short time occupied by the collision the vector-potential must be changing very rapidly. But any change in the vector-potential produces a corresponding electromotive force, and thus the glass against which the molecules impinge is subjected to a rapidly varying electromotive force. But this, if Maxwell's electromagnetic theory of light be true, is exactly what it is subjected to when a beam of light falls upon it, which we know is the ordinary method of exciting phosphorescence. Stokes's law, that the period of the vibrations exciting the phosphorescence is smaller than the period of the emitted light, compels us to assume that at some period of the collision the velocity of the moving particle is changing at a greater rate than the rate of vibration of green light: in our present state of knowledge, however, there seems nothing impossible in this. This, too, would explain the following difficulty:—Since we have every reason for supposing the discharge in a vacuum-tube to be discontinuous, the vector-potential due to electricity moving through the tube will vary, producing a varying electromotive force all over the tube; another varying electromotive force will be produced by the action of the charge on the electrodes. Now it may be asked, why, if the above theory be true, does not this variable electromotive force make the whole tube phosphoresce, instead of the phosphorescence being confined to the places where the molecular streams strike the glass. But Spottiswoode and Moulton have proved (see Phil. Trans. for 1879, part 2) that the time occupied by the negative discharge is greater than the time occupied by the particles in going the length of the tube. Hence, even if we made the extravagant assumption that these molecules travel with a velocity as great as that of light, the time of discharge, and consequently the period of the electromotive force, would be greater than the period of vibration of light whose wave-length was the length of the tube, and so, by Stokes's law, could not produce a luminous phosphorescence.

It may be useful to form a rough estimate of the electromotive force which could be produced by a moving particle.

By equation (1) we see, if the particle be moving parallel to the axis of x with velocity p, that the greatest value of F at a point distant R from the centre of the particle is

$$\mu p\left(\frac{1}{\mathrm{R}}-\frac{a^2}{5\mathrm{R}^3}\right).$$

Now the greatest value of e, as before, is $\mathrm{K}\times 3\times 10^{12}\times a^2$,

$$\mu\mathrm{K}=\frac{1}{9\times 10^{20}};$$

hence the greatest value of F at the surface of the particle

$$=\frac{3\times 4\times 10^{12}pa}{5\times 9\times 10^{20}}.$$

Now during the collision let us represent p by $p_0\cos kr$, where $\frac{2\pi}{k}$ is less than the period of vibration of green light; R must be therefore at least 3×10^{15}; for a particle of air a is of the order 10^{-7}. Substituting, we get

$$\frac{d\mathrm{F}}{dt}=-\frac{4\times 10^5}{15\times 10^{20}}\mathrm{R}p_0\sin \mathrm{R}t;$$

or the maximum value of $\frac{d\mathrm{F}}{dt}$ is

$$\frac{4\times 3\times 10^{20}}{15\times 10^{20}}p^0=\tfrac{4}{5}p_0.$$

Now at present we know nothing about p_0; but it must be very much greater than the mean velocity of the air-molecules, which is about 5×10^4; if we substitute this value for it, we get the maximum value of $\frac{d\mathrm{F}}{dt}$ or the maximum electromotive force to be about 4×10^4, or about $\frac{1}{2500}$ of a volt per centimetre. Now, for sunlight the maximum electromotive force is about 6 volts per centimetre (Maxwell's 'Electricity and Magnetism,' § 793); and when we consider the immense number of particles which must be striking the glass at each instant, we have no difficulty in conceiving that the magnitude of the electromotive force due to the moving particle may be sufficient to cause phosphorescence. To show the rapidity with which these electromotive forces diminish with the distance, we will take the case of a particle stopped by a screen at a distance of $\frac{1}{100}$ of a millimetre from the glass, and compare the electromotive force at the glass with the electromotive force which would be produced at the glass if there were no screen. By substituting in the formula giving the electromotive force, we find that the electromotive force at the glass when the screen is present is only about $\frac{1}{10000}$ of what it is when the screen is away; and as the intensity of the phosphorescence will vary as the square of the electromotive force, we see that when the screen is present the phosphorescence is quite imperceptible. This explains an experiment of Goldstein's, in which he coated the glass with a layer of collodion whose thickness he estimated at a few hundredths of a millimetre, and found the glass behind quite black.

§ 5. To find the effect produced by a magnet on a moving electrified sphere. To do this we shall calculate the kinetic energy of the system; we can then, by means of Lagrange's equations, calculate the force on the sphere.

Let α, β, γ be the components of magnetic induction, α_1, β_1, γ_1 the components of magnetic force; if A, B, C be the components of magnetization,

$$\alpha=\alpha_1+4\pi\mathrm{A}, \quad \beta=\beta_1+4\pi\mathrm{B}, \quad \gamma=\gamma_1+4\pi\mathrm{C}.$$

The kinetic energy of the system

$$=\frac{1}{8\pi}\iiint(\alpha\alpha_1+\beta\beta_2+\gamma\gamma_1)dx\,dy\,dz.$$

To get the force on the sphere due to the magnet, we only want that part of the kinetic energy which involves both the coordinates of the sphere and the coordinates of the magnet. We may write the kinetic energy as

$$\frac{1}{8\pi}\iiint(\alpha^2+\beta^2+\gamma^2-4\pi\alpha\mathrm{A}-4\pi\beta\mathrm{B}-4\pi\gamma\mathrm{C})dx\,dy\,dz.$$

Let F′, G′, H′ be the components of the vector-potential due to the magnet alone; then, by equation (4),

$$\left.\begin{aligned}\alpha&=\mu e\left(r\frac{d}{dy}\frac{1}{\mathrm{R}}-q\frac{d}{dz}\frac{1}{\mathrm{R}}\right)+\frac{d\mathrm{H}'}{dy}-\frac{d\mathrm{G}'}{dz},\\ \beta&=\mu e\left(p\frac{d}{dz}\frac{1}{\mathrm{R}}-r\frac{d}{dx}\frac{1}{\mathrm{R}}\right)+\frac{d\mathrm{F}'}{dz}-\frac{d\mathrm{H}'}{dx},\\ \gamma&=\mu e\left(q\frac{d}{dx}\frac{1}{\mathrm{R}}-p\frac{d}{dy}\frac{1}{\mathrm{R}}\right)+\frac{d\mathrm{G}'}{dx}-\frac{d\mathrm{F}'}{dy}.\end{aligned}\right\}$$

The part of the kinetic energy we are concerned with will evidently be

$$\frac{1}{8\pi}\iiint 2\mu e\left[\left(r\frac{d}{dy}-q\frac{d}{dz}\right)\frac{1}{\mathrm{R}}\left(\frac{d\mathrm{H}'}{dy}-\frac{d\mathrm{G}'}{dz}\right)\right.$$

$$+\left(p\frac{d}{dz}-r\frac{d}{dx}\right)\frac{1}{\mathrm{R}}\left(\frac{d\mathrm{F}'}{dz}-\frac{d\mathrm{H}'}{dx}\right)$$

$$\left.+\left(q\frac{d}{dx}-p\frac{d}{dy}\right)\frac{1}{\mathrm{R}}\left(\frac{d\mathrm{G}'}{dx}-\frac{d\mathrm{F}'}{dy}\right)\right]dx\,dy\,dz$$

$$-\frac{1}{8\pi}\iiint\mu e 4\pi\left[\mathrm{A}\left(r\frac{d}{dy}-q\frac{d}{dz}\right)\frac{1}{\mathrm{R}}+\mathrm{B}\left(p\frac{d}{dz}-r\frac{d}{dx}\right)\frac{1}{\mathrm{R}}\right.$$

$$\left.+\mathrm{C}\left(q\frac{d}{dx}-p\frac{d}{dy}\right)\frac{1}{\mathrm{R}}\right]dx\,dy\,dz.$$

Let us take the first integral first, and take the term depending on p; this is

$$\frac{\mu ep}{4\pi}\iiint\frac{d}{dz}\frac{1}{\mathrm{R}}\left(\frac{d\mathrm{F}'}{dz}-\frac{d\mathrm{H}'}{dx}\right)-\frac{d}{dy}\frac{1}{\mathrm{R}}\left(\frac{d\mathrm{G}'}{dx}-\frac{d\mathrm{F}'}{dy}\right)dx\,dy\,dz.$$

Integrating by parts this becomes

$$-\frac{\mu ep}{4\pi}\iint\mathrm{F}'\left(\frac{d}{dx}\frac{1}{\mathrm{R}}dy\,dz+\frac{d}{dy}\frac{1}{\mathrm{R}}dx\,dz+\frac{d}{dz}\frac{1}{\mathrm{R}}dx\,dy\right)$$

$$+\frac{\mu ep}{4\pi}\iint\frac{1}{\mathrm{R}}\left(\frac{d\mathrm{H}'}{dx}dy\,dx+\frac{d\mathrm{G}'}{dx}dx\,dz+\frac{d\mathrm{F}'}{dx}dz\,dy\right)$$

$$+\frac{\mu ep}{4\pi}\iiint\frac{1}{\mathrm{R}}\frac{d}{dx}\left(\frac{d\mathrm{F}'}{dx}+\frac{d\mathrm{G}'}{dy}+\frac{d\mathrm{H}'}{dz}\right)$$

$$-\mathrm{F}'\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\frac{1}{\mathrm{R}}dx\,dy\,dz.$$

The surface-integrals are to be taken over the surface of the sphere; and the triple integral is to be taken throughout all space exterior to the sphere.

If the sphere be so small that we may substitute for the values of F', $\frac{d\mathrm{F}'}{dx}$, &c. at the surface their values at the centre of the sphere, the first surface-integral $=\mu ep\mathrm{F}'_1$, where F'_1 is the value of F' at the centre of the sphere; the second surface-integral vanishes, and the triple integral also vanishes, since

$$\frac{d^2}{dx^2}\frac{1}{\mathrm{R}}+\frac{d^2}{dy^2}\frac{1}{\mathrm{R}}+\frac{d^2}{dz^2}\frac{1}{\mathrm{R}}=0,$$

and

$$\frac{d\mathrm{F}'}{dx}+\frac{d\mathrm{G}'}{dy}+\frac{d\mathrm{H}'}{dz}=0.$$

The part depending on p in the second integral

$$=-\frac{\mu ep}{2}\iiint\left(\mathrm{B}\frac{d}{dz}\frac{1}{\mathrm{R}}-\frac{\mathrm{C}d}{dy}\frac{1}{\mathrm{R}}\right)dx\,dy\,dz,$$

or (see Maxwell's 'Electricity and Magnetism,' § 405)

$$=-\frac{\mu ep}{2}\mathrm{F}'_1.$$

Adding this to the term $\mu ep\mathrm{F}'_1$ already obtained, we get $\frac{\mu ep}{2}\mathrm{F}'_1$ as the part of the kinetic energy depending on p. We have evidently similar expressions for the parts of the kinetic energy depending on q and r. Hence the part of the kinetic energy with which we are concerned will

$$=\frac{\mu e}{2}\cdot(\mathrm{F}'_1 p+\mathrm{G}'_1 q+\mathrm{H}'_1 r).$$

By Lagrange's equations, the force on the sphere parallel to the axis of x

$$=\frac{d\mathrm{T}}{dx}-\frac{d}{dt}\frac{d\mathrm{T}}{d\dot{x}}$$

$$=\frac{\mu e}{2}\left\{p\frac{d\mathrm{F}'_1}{dx}+q\frac{d\mathrm{G}'_1}{dx}+r\frac{d\mathrm{H}'_1}{dx}-\frac{d\mathrm{F}'_1}{dt}\right\}$$

$$=\frac{\mu e}{2}\left(p\frac{d\mathrm{F}'_1}{dx}+q\frac{d\mathrm{G}'_1}{dx}+r\frac{d\mathrm{H}'_1}{dx}-p\frac{d\mathrm{F}'_1}{dx}-q\frac{d\mathrm{F}'_1}{dy}-r\frac{d\mathrm{F}'_1}{dz}\right)\}$$

$$=\frac{\mu e}{2}\left\{q\left(\frac{d\mathrm{G}'_1}{dx}-\frac{d\mathrm{F}'_1}{dy}\right)-r\left(\frac{d\mathrm{F}'_1}{dz}-\frac{d\mathrm{H}'_1}{dx}\right)\right\}$$

$$=\frac{\mu e}{2}(qc_1-rb_1).$$

Similarly, the force parallel to the axis of y

$$=\frac{\mu e}{2}(ra_1-pc_1), \quad \ldots\ldots \quad (5)$$

the force parallel to the axis of z

$$=\frac{\mu e}{2}(pb_1-qa_1),$$

where a_1, b_1, c_1 are the components of magnetic induction at the centre of the sphere due to the external magnet. These forces are the same as would act on unit length of a conductor at the centre of the sphere carrying a current whose components are $\frac{\mu ep}{2}$, $\frac{\mu eq}{2}$, $\frac{\mu er}{2}$. The resultant force is perpendicular to the direction of motion of the sphere and to the magnetic induction; and if ω be the resultant velocity of the sphere, and θ the angle between the direction of motion of the sphere and the direction of magnetic induction, the magnitude of the force

$$=\frac{\mu e}{2}\omega\sqrt{a^2+b^2+c^2}\sin\theta.$$

It will be useful to endeavour to calculate the magnitude of this force on a particle of air moving in a vacuum-tube; although our knowledge of the magnitude of several of the quantities involved is so vague that our result must only be looked upon as showing that the force is of an order great enough to produce appreciable effects, and must not be looked upon as having any quantitative value.

Let us suppose that the mass of a molecule of air is 10^{-22} (C.G.S. system); that a, the radius of the molecule, $=10^{-7}$; that, as before, $e=\mathrm{K}\times 3\times 10^{12}a^2=\mathrm{K}\times 3\times 10^{-2}$ (this quantity is probably enormously underrated); and as we know nothing about the velocity of the charged particles, let us assume it to be the mean velocity of the air-molecules, viz. 4×10^{-5}. We shall suppose the vacuum-tube placed in a magnetic field whose strength is 10^3. Then, by the formula, the acceleration of the particle of air when the magnetic force is at right angles to its path is about 10^7; this acceleration would produce a deflection of about 2 millims. per decimetre of path, a deflection which could easily be observed. We know from the experiments of Mr. Crookes and others that a magnet produces very decided deflections of the molecular streams; and the direction of the deflections (see Phil. Trans. 1879, part 1, pp. 154 & 156) agrees with that given by formulæ (5), if we suppose that the particles projected from the negative pole are negatively charged.

§ 6. Let us now calculate the expression given by Maxwell's theory for the force between two charged moving particles.

Let u, v, w be the components of the velocity of the centre of one of the particles, u', v', w' those of the other; let R denote the distance between the particles, e the charge on one of the particles, e' the charge on the other; let r denote the distance of a point from the centre of the first particle, r' the distance of the same point from the centre of the second particle. We shall suppose, for the sake of simplicity, that the particles are very small; we shall calculate the kinetic energy of the system and deduce the forces between the particles by means of Lagrange's equations.

The kinetic energy

$$=\tfrac{1}{2}\iiint\left(\mathrm{F}\frac{df}{dt}+\mathrm{G}\frac{dg}{dt}+\mathrm{H}\frac{dh}{dt}\right)dx\,dy\,dz.$$

Now

$$\mathrm{F}=\frac{\mu}{5}\left[e\left(u\frac{d^2}{dx^2}\frac{1}{\mathrm{R}}+v\frac{d^2}{dx\,dy}\frac{1}{r}+w\frac{d^2}{dx\,dz}\frac{1}{r}\right)\left(\frac{5r^2}{6}-\frac{a^2}{2}\right)+e\frac{10}{3}\frac{u}{r}\right.$$

$$\left.+e'\left(u'\frac{d^2}{dx^2}\frac{1}{r'}+v'\frac{d^2}{dx\,dy}\frac{1}{r'}+w'\frac{d^2}{dx\,dz}\frac{1}{r'}\right)\left(\frac{5r'^2}{6}-\frac{a'^2}{2}\right)+\frac{e'10}{3}\frac{u'}{r'}\right],$$

with similar expressions for G and H.

$$\frac{df}{dt}=\frac{1}{4\pi}\left[e\left(u\frac{d^2}{dx^2}\frac{1}{r}+v\frac{d^2}{dx\,dy}\frac{1}{r}+w\frac{d^2}{dx\,dz}\frac{1}{r}\right)\right.$$

$$\left.+e'\left(u'\frac{d^2}{dx^2}\frac{1}{r'}+v'\frac{d^2}{dx\,dy}\frac{1}{r'}+w'\frac{d^2}{dx\,dz}\frac{1}{r'}\right)\right],$$

with similar expressions for $\frac{dg}{dt}$ and $\frac{dh}{dt}$. Since the particles are supposed to be very small, we shall neglect those terms in F which depend on a^2 and a'^2.

The part of the kinetic energy we are concerned with involves the product $e\,e'$: let us first calculate that part of it arising from the product of that part of F due to e with that part of $\frac{df}{dt}$ due to e'. We shall take the line joining the particle as the axis of x; and for brevity we shall denote $\frac{\mu e e'}{24\pi}$ by σ.

The coefficient of $u\,u'$ in the part of the kinetic energy we are considering

$$=\sigma\iiint\left(\frac{d^2}{dx^2}\frac{1}{r}+\frac{4}{r^3}\right)r^2\frac{d^2}{dx^2}\frac{1}{r'}\,dx\,dy\,dz.$$

Now, for values of $r>\mathrm{R}$,

$$\frac{1}{r'}=\frac{1}{r}-\mathrm{R}\frac{d}{dx}\frac{1}{r}+\frac{\mathrm{R}^2}{2\,!}\frac{d^2}{dx^2}\frac{1}{r}-\ldots;$$

$$\therefore\ \frac{d^2}{dx^2}\frac{1}{r'}=\frac{d^2}{dx^2}\frac{1}{r}-\mathrm{R}\frac{d^3}{dx^3}\frac{1}{r}+\ldots.$$

Now, since

$$\frac{d^n}{dx^n}\frac{1}{r}=(-)^n\frac{n\,!}{r^{n+1}}\mathrm{Q}_n,$$

where Q_n is a zonal harmonic of the nth order; and since the product of two harmonics of different degrees integrated over the surface of a sphere vanishes, we may substitute in the integral $\frac{d^2}{dx^2}\frac{1}{r}$ for $\frac{d^2}{dx^2}\frac{1}{r'}$; then, transforming to polars, the integral

$$=\sigma\int_0^{2\pi}\int_0^{\pi}\int_{R}^{\infty} 4Q_2^2\frac{1}{r^2}\sin\theta\, d\phi\, dv\, dr$$

$$=\frac{16\pi\sigma}{5R};$$

for values of $r<R$,

$$\frac{1}{r'}=\frac{1}{R}+\frac{rQ_1}{R^2}+\frac{rQ_2}{R^3}+\ldots.$$

Now $r^n Q_n$ is a solid harmonic of the nth order; hence $\frac{d^2}{dx^2}(r^n Q_n)$ is a solid harmonic of the $(n-2)$th order; and in particular $\frac{d^2}{dx^2}(r^4Q_4)$ is a solid harmonic of the second order; and, by the same reasoning as before, we may substitute in the integral $\frac{1}{R^5}\frac{d^2}{dx^2}(r^4Q_4)$ for $\frac{d^2}{dx^2}\frac{1}{r'}$. Now

$$r^4Q_4=\frac{35x^4-30x^2(x^2+y^2+z^2)+3(x^2+y^2+z^2)^2}{8};$$

$$\therefore\ \frac{d^2}{dx^2}(r^4Q_4)=12x^2-6(y^2+z^2)=12r^2Q_2.$$

So for values of $r<R$ the integral becomes

$$\frac{\sigma}{R^5}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} 24Q_2^2 r^3\sin\theta\, d\phi\, d\theta\, dr$$

$$=\frac{24\sigma\pi}{5R}.$$

Adding this to the part of the integral for $r>R$, we get for the coefficient of uu', $\frac{8\pi\sigma}{R}$. The coefficients of uv' and uw' vanish by inspection.

The coefficient of vv'

$$=\sigma\iiint r^2\frac{d^2}{dx\,dy}\frac{1}{r}\ \frac{d^2}{dx\,d}\frac{1}{r'}\,dx\,dy\,dz.$$

Now when $r>R$ we may, by the same reasoning as before, substitute $\frac{d^2}{dx\,dy}\frac{1}{r}$ for $\frac{d^2}{dx\,dy}\frac{1}{r'}$ in the integral, and it becomes

$$\sigma\iiint\frac{9r^2x^2y^2}{r^{10}}\,dx\,dy\,dz,$$

or transforming to polars,

$$9\sigma\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{\sin^2\theta\cos^2\theta\cos^2\phi}{r^2}\sin\theta\,d\phi\,d\theta\,dr$$
$$=\frac{36\pi\sigma}{15R}.$$

For values of $r<R$ we may, by the same reasoning as before, substitute $\frac{1}{R^5}\frac{d^2}{dx\,dy}(r^4Q_4)$ in the integral for $\frac{d^2}{dx\,dy}\frac{1}{r'}$. Now

$$\frac{d^2}{dx\,dy}(r^4Q_4)=-12xy\,;$$

making this substitution, the integral becomes

$$-\frac{\sigma}{R^5}\iiint\frac{36x^2y^2}{r^5}\,dx\,dy\,dz$$
$$=-\frac{\sigma}{R^5}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}36r^3\sin^3\theta\cos^2\theta\cos^2\phi\,d\phi\,d\theta\,dr$$
$$=-\frac{36\pi\sigma}{15R}.$$

Hence, adding this to the part previously obtained for values of $r>R$, we see that the coefficient of vv' from $F\frac{df}{dt}$ is zero, and, similarly, the coefficient of ww' from this part of the integral vanishes.

Let us now take the terms arising from $\iiint G\frac{dg}{dt}\,dx\,dy\,dz$, and take, as before, the part arising from the product of that part of G due to e with the part of $\frac{dg}{dt}$ due to e'. The coefficient of uu' in this part will be the same as the coefficient of vv' in the former part, and so will vanish.

The coefficient of vv'

$$=\sigma\iiint\left(r^2\frac{d^2}{dy^2}\frac{1}{r}+\frac{4}{r}\right)\frac{d^2}{dx^2}\frac{1}{r'}\,dx\,dy\,dz.$$

Now for values of $r>R$ we may, as before, substitute $\frac{d^2}{dy^2}\frac{1}{r}$ for $\frac{d^2}{dy^2}\frac{1}{r'}$; and it becomes

$$\sigma\iiint r^2\left(\frac{3y^2-r}{r}\right)^2dx\,dy\,dz.$$

By transforming to polars, as before, this may be shown to be

$\frac{16\sigma\pi}{5}$. For values of $r<\mathrm{R}$ we may, as before, substitute $\frac{1}{\mathrm{R}^5}\frac{d^2}{dy^2}(r^4\mathrm{Q}_4)$ for $\frac{d^2}{dy^2}\frac{1}{r'}$ in the integral. Now

$$\frac{d^2}{dy^2}(r^4\mathrm{Q}_4)=\frac{36y^2+12z^2-48x^2}{8};$$

∴ the integral

$$=\frac{\sigma}{\mathrm{R}^5}\iiint\frac{r^2(3y^2-r^2)(36y^2+12z^2-48x^2)}{8r^5}\,dx\,dy\,dz.$$

By transforming to polars, this may be shown to be $\frac{9\pi\sigma}{5\mathrm{R}}$. Adding this to the part of the integral due to values of $r>\mathrm{R}$, we get for the coefficient of vv',

$$\frac{5\sigma\pi}{\mathrm{R}}.$$

As before, the coefficients of uv', vu', uw', &c. disappear by inspection.

The coefficient of ww'

$$=\sigma\iiint r^2\frac{d^2}{dy\,dz}\frac{1}{r}\,\frac{d^2}{dy\,dz}\frac{1}{r'}\,dx\,dy\,dz;$$

substituting, for values of $r>\mathrm{R}$, as before $\frac{d^2}{dy\,dz}\frac{1}{r}$ for $\frac{d^2}{dy\,dz}\frac{1}{r'}$ in the integral, it becomes

$$\sigma\iiint\frac{9y^2z^2}{r^8}\,dx\,dy\,dz,$$

which, by transforming to polars, may be shown to be $\frac{12\sigma\pi}{5\mathrm{R}}$. For values of $r<\mathrm{R}$ we may, as before, substitute $\frac{1}{\mathrm{R}^5}\frac{d^2}{dy\,dz}(r^4\mathrm{Q}_4)$ for $\frac{d^2}{dy\,dz}\frac{1}{r'}$ in the integral. Now

$$\frac{d^2}{dy\,dz}(r^4\mathrm{Q}_4)=3yz.$$

On making this substitution, the integral

$$=\frac{\sigma}{\mathrm{R}^5}\iiint\frac{9y^2z^2}{r^3}\,dx\,dy\,dz=\frac{3\sigma\pi}{5\mathrm{R}}.$$

Adding this to the part obtained before, we get for the coefficient of ww',

$$\frac{12\sigma\pi}{5\mathrm{R}}+\frac{3\sigma\pi}{5\mathrm{R}},\text{ or }3\sigma\pi.$$

From the part of $\iiint\mathrm{H}\frac{dh}{dt}\,dx\,dy\,dz$ which arises from that part of H due to e and that part of $\frac{dh}{dt}$ due to e', we can see, by the preceding work, that the coefficient of uu' is zero; the coefficient of vv', $3\sigma\pi$; and the coefficient of ww', $5\sigma\pi$. Adding, we get the whole kinetic energy due to the vector-potential arising from e and the electric displacement arising from e'

$$=\frac{\pi\sigma}{2R}\left(8uu'+(5+3)vv'+(5+3)ww'\right)$$

$$=\frac{4\pi\sigma}{R}(uu'+vv'+ww').$$

We can get that part of the kinetic energy due to the vector-potential arising from e' and the electric displacement from e by writing e' for e, and u', v', w' for u, v, w respectively. Hence, that part of the kinetic energy which is multiplied by ee'

$$=\frac{8\pi\sigma}{R}(uu'+vv'+ww');$$

or, substituting for σ its value,

$$=\frac{\mu ee'}{3R}(uu'+vv'+ww').$$

Or if q and q' be the velocities of the spheres, and ϵ the angle between their directions of motion, this part of the kinetic energy

$$=\frac{\mu ee'}{3R}qq'\cos\epsilon,$$

and the whole kinetic energy due to the electrification

$$=\mu\left(\frac{2}{15}\frac{e^2q^2}{a}+\frac{2}{15}\frac{e'^2q'^2}{a_1}+\frac{ee'}{3R}qq'\cos\epsilon\right). \quad . \quad . \quad (6)$$

If x, y, z be the coordinates of the centre of one sphere, x', y', z' those of the other, we may write the last part of the kinetic energy in the form

$$\frac{\mu ee'}{3R}\left(\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\frac{dz}{dt}\frac{dz'}{dt}\right).$$

By Lagrange's equations, the force parallel to the axis of x acting on the first sphere

$$=\frac{dT}{dx}-\frac{d}{dt}\left(\frac{dT}{d\cdot\frac{dx}{dt}}\right)$$

$$=\frac{\mu ee'}{3}\left\{\left(\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\frac{dz}{dt}\frac{dz'}{dt}\right)\frac{d}{dx}\frac{1}{R}-\frac{d}{dt}\left(\frac{\frac{dx'}{dt}}{R}\right)\right\},$$

with similar expressions for the components of the force parallel to the axes of y and z.

From this we see that if $\dot{q}_1$ be the acceleration of the second sphere, the forces on the first sphere are an attraction $\frac{\mu ee'}{3R^2}qq'\cos\epsilon$ along the line joining the centres of the spheres, a force $\frac{\mu ee'}{3R}\dot{q}_1$ in the direction opposite to the acceleration of the second sphere, and a force $\frac{\mu ee'}{3}q_1\frac{d}{dt}\left(\frac{1}{R}\right)$ in the direction opposite to the direction of motion of the second sphere. There are, of course, corresponding forces on the second sphere; and we see that, unless both spheres move with equal uniform velocities in the same direction, the forces on the two spheres are not equal and opposite. If we suppose that the two spheres are moving with uniform velocities q in the same direction, the repulsion between them is $\frac{ee'}{K.R^2}\left(1-\frac{\mu Kq^2}{3}\right)$; or if c be the velocity of light in the medium through which they are moving, the repulsion $=\frac{ee'}{KR^2}\left(1-\frac{q^2}{3c^2}\right)$. Hence, if the repulsion between two electrified particles is to be changed into an attraction by means of their motion, their velocities must exceed $\sqrt{3}c$; hence we should expect the molecular streams in a vacuum-tube to repel each other, as we could not suppose that the velocity of the particles forming these streams is as great as that of light; and Mr. Crookes has, in fact (see Phil. Trans. 1879, part ii.), experimentally determined that they do repel each other.

It is remarkable that the law of force between two moving charged particles, which we have deduced from Maxwell's theory, agrees with that assumed by Clausius, in his recent researches on Electrodynamics (see Phil. Mag. Oct. 1880); but it differs from Weber's well-known law materially. According to Weber's law, the force does not depend on the actual velocities of the particles, but only on their velocity relative to each other, whereas, according to the laws we have investigated, the forces depend on the actual velocities of the particles as well as on their relative velocities: thus there is a force between two charged particles moving with equal velocities in the same direction, in which case, of course, the relative velocity is nothing. It must be remarked that what we have for convenience called the actual velocity of the particle is, in fact, the velocity of the particle relative to the medium through which it is moving: thus, in equation (6), q, q' are the velocities of the first and second particles respectively relative to the medium whose magnetic permeability is μ.

Clausius, in the paper previously referred to, explains the various phenomena produced by currents by means of this law of force, and the hypothesis that a current consists of streams of opposite electricities moving in opposite directions. Now, since the expressions we have obtained for the force between the particle do not depend on the specific inductive capacity of the medium, but only on its magnetic permeability, if we make this assumption about the nature of a current, it follows from Maxwell's theory that the electrodynamic phenomena produced by a current of given strength do not depend on the specific inductive capacity of the surrounding medium, though they do depend on its magnetic permeability.

Faraday, in his 'Experimental Researches' (§ 1709 and onwards), describes some experiments which he made to determine whether altering the surrounding medium produced any change in the electromagnetic action of a current. The result of the experiments was that he was unable to detect any such change; but in his experiments, though the specific inductive capacities of the various media tried were very different, their magnetic permeabilities were all of them very nearly unity.

XXXIV. *Theoretical Explanations of the Rectilinear Transmission and Spontaneous Diffusion of Sound and Light.* *By* Professor CHALLIS, M.A., F.R.S., F.R.A.S.*

WHEN any disturbance is produced at a given position A in an unlimited mass of elastic fluid of perfect fluidity, defined by the relation $p=a^2\rho$ between its pressure p and density ρ, a^2 being constant, it is found by experience that there will be a resulting state of the fluid at a point P, whose position is taken *ad libitum*, and at all intermediate points between A and P. In other words, there will be a rectilinear transmission of effect from A to all surrounding points, without *respect to the particular mode of disturbing the fluid.* It may be that a difference of effect in different directions may depend on the mode of disturbance; at the same time it is found that a resulting disturbance is produced at *all* points, whatever be the mode of disturbance. In treatises on hydrodynamics this remarkable fact is left out of consideration. I know of none in which this problem has been solved, or even proposed. But it is an admitted principle that when the *fundamentals* of any branch of applied science, after being established by observation and experiment, have been ex-

* Communicated by the Author.