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YAMABE SOLITONS AND τ-QUASI YAMABE GRADIENT SOLITONS ON RIEMANNIAN MANIFOLDS ADMITTING CONCURRENT-RECURRENT VECTOR FIELDS

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ABSTRACT. We consider a Riemannian manifold (M, g) admitting a concurrent-recurrent vector field for which the metric g is a Yamabe soliton or a τ-quasi Yamabe gradient soliton. We show that if the metric of a Riemannian three-manifold (M, g) admitting a concurrent-recurrent vector field is a Yamabe soliton, then M is of constant negative curvature $-\alpha^2$. In this case, we see that the potential vector field is Killing. Next, we show that if the metric of a Riemannian manifold M admitting concurrent-recurrent vector field is a non-trivial τ-quasi Yamabe gradient soliton with potential function f, then M has constant scalar curvature and is equal to $-n(n-1)\alpha^2$. Finally, an illustrative example is presented.

1. Introduction

Let (M, g) be a compact Riemannian manifold with s as its scalar curvature. Then, the Yamabe problem concerns the existence of a Riemannian metric g' conformal to g, for which the scalar curvature s' of the metric g' is constant. As an effort to solve the Yamabe problem, Hamilton in [4] came up with the concept of Yamabe flow. Given a conformal class of Riemannian metrics, Yamabe flow can be used for constructing metrics whose scalar curvature s is constant. The Yamabe flow is an evolving metric family $(g(t))$ satisfying

$$\frac{\partial}{\partial t} g(t) = -s(t) g(t) \tag{1.1}$$

with the initial data $g(0) = g$. Geometric flows like Yamabe flows, Ricci flows, mean and the inverse mean curvature flows, and Kaehler-Ricci flows have been applied to a variety of topological, geometric, and physical problems. It is interesting to note that in dimension two the Yamabe flow and Ricci flow are equivalent, but in higher dimension they are non-identical.

The Yamabe solitons are special solutions of the Yamabe flows, that is, there exist scalars $\sigma(t)$ and diffeomorphisms ϕ_t in such manner that $g(t) = \sigma(t)\phi_t^*(g_0)$ is the solution of the Yamabe flow (1.1), with $\sigma(0) = 1$ and $\phi_0 = I$. In other words, a Riemannian metric g is called a Yamabe soliton if there exist a smooth vector field $V \in \mathfrak{X}(M)$ (called a potential vector field) and a scalar $\lambda \in \mathbb{R}$ such that

$$\frac{1}{2}\mathcal{L}_V g = (s - \lambda) g, \tag{1.2}$$

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where V stands for the Lie derivative operator along V. In the case which V is Killing, we say that the Yamabe soliton is *trivial*. In particular, if the potential vector field $V = Df$, where D is the gradient operator and f is a smooth scalar function, then we say that the metric g is *Yamabe gradient soliton* and in this case (1.2) turns into

$$\mathrm{Hess}_f = (s - \lambda)g.$$

where Hess_f is Hessian of f. If f is constant, then the above soliton called *trivial* Yamabe gradient soliton.

It is interesting to notice that there is a nice connection between warped product structure and Yamabe soliton. In [6], Ma and Cheng showed that a non-compact complete Riemannian manifold admitting a Yamabe gradient soliton has a warped product structure. In [10], Tokura et al. studied Yamabe gradient soliton on warped product manifold with compact Riemannian base and in that case it has been shown that the soliton is trivial. Yamabe solitons have been studied by many geometers in many different contexts (see [1–3, 8, 9, 11, 13]). Recently, in [7], the first author introduced a special vector field ν which satisfies the relation

$$\nabla_X \nu = \alpha\{X - \nu^\flat(X)\nu\}, \tag{1.3}$$

where $\alpha \in \mathbb{R}$ and $\nu^\flat$ is the 1-form equivalent to ν in a Reimannian manifold (M, g). A unit non-parallel (i.e., $\alpha \neq 0$) vector field ν satisfying the preceding equation is called a *concurrent-recurrent vector field*. It is shown that an n-dimensional connected Riemannian manifold (M, g) admits concurrent-recurrent vector field ν, if and only if, (M, g) is the warped product $I \times_{f(t)} F$, where I is an open interval and $f(t) = \mathrm{e}^{\alpha t}$ (see [7: Theorem 3]). In this paper, we consider a Yamabe soliton on a Riemannian manifold admitting concurrent-recurrent vector field and prove:

Theorem 1.1. *Let (M, g) be Riemannian three-manifold admitting concurrent-recurrent vector field. If the metric g is a Yamabe soliton with potential vector field V, then the manifold is of constant negative curvature $-\alpha^2$ and V is Killing.*

Due to the Theorem 3 of [7], we have:

Corollary 1.1.1. *Let $M = I \times_{f(t)} F$ with the warping function $f(t) = \mathrm{e}^{\alpha t}$, where $\alpha \in \mathbb{R}$, I is an open interval in $\mathbb{R}$ and F is a Riemannian 2-manifold. If the metric of M is a Yamabe soliton, then the manifold is of constant negative curvature $-\alpha^2$.*

In [5], Huang and Li introduced the notion of a τ-quasi Yamabe gradient soliton which naturally extends the concept of Yamabe gradient soliton. According to Huang and Li [5], a τ-quasi Yamabe gradient soliton is a Riemannian metric g satisfying

$$\mathrm{Hess}_f = \frac{1}{\tau}\mathrm{d}f \otimes \mathrm{d}\mathrm{d}f + (s - \lambda)g, \tag{1.4}$$

where f is a smooth scalar function and $\tau > 0$ is a constant. Notice that a Yamabe gradient soliton is nothing but an ∞-quasi Yamabe gradient soliton. For $\lambda < 0$ the Yamabe soliton (or τ-quasi Yamabe gradient soliton) is said to be shrinking, for $\lambda > 0$ is said to be expanding, and for $\lambda = 0$ is said to be steady. In [12], Wang showed that a non-compact complete Riemannian manifold admitting a τ-quasi Yamabe gradient soliton has warped product structure. In this direction, we consider a τ-quasi Yamabe gradient soliton on a Riemannian manifold admitting concurrent-recurrent vector field and prove the following theorem.

Theorem 1.2. *Let M be Riemannian manifold admitting concurrent-recurrent vector field. If the metric of M is non-trivial τ-quasi Yamabe gradient soliton with potential function f, then the scalar curvature of M is constant and is equal to $-n(n-1)\alpha^2$.*

From Theorem 3 of [7], we immediately have the following.

Corollary 1.2.1. *Let $M = I \times_{f(t)} F$ with the warping function $f(t) = \mathrm{e}^{\alpha t}$, where $\alpha \in \mathbb{R}$, I is an open interval in $\mathbb{R}$ and F is a Riemannian n-manifold. If the metric of M is a non-trivial τ-quasi Yamabe gradient soliton, then the scalar curvature of M is constant and is equal to $-n(n-1)\alpha^2$.*

2. Background and key lemmas

A unit vector field ν on a Reimannian manifold (M, g) is said to be a concurrent-recurrent vector field if it satisfies

$$\nabla_X \nu = \alpha\{X - \nu^\flat(X)\nu\}, \tag{2.1}$$

where ∇ is the Levi-Civita connection of g and α is a non-zero constant. In [7], the author constructed certain examples of n-dimensional Riemannian manifolds admitting such vector fields. An interesting property of this vector field is that it is an eigenvector of the Ricci operator of the Riemannian manifold (M, g) on which this vector field is defined. Moreover, the defining equation (2.1) dictates that integral curves of ν are geodesics. The following result has been proved in [7].

Theorem 2.1. *A Riemannian n-manifold admitting a concurrent-recurrent vector field is locally isometric to the warped product $I \times_{f(t)} F$, where $I \subseteq \mathbb{R}$ is an open interval and F is a Riemannian $(n-1)$-manifold. Conversely, the warped product $I \times_{f(t)} F$ with the warping function $f(t) = \mathrm{e}^{\alpha t}$ admits a concurrent-recurrent vector field.*

2.1. Key lemmas

In this subsection, we give some lemmas that are needed to prove our main results.

Lemma 2.1. *A Riemannian manifold equipped with a concurrent-recurrent vector field ν satisfies*

$$\nu(s) = -2\alpha(s + n(n-1)\alpha^2). \tag{2.2}$$

Proof. Using (1.3) in the definition of Riemann curvature tensor, we obtain

$$R(X, Y)\nu = -\alpha^2\{\nu^\flat(Y)X - \nu^\flat(X)Y\}. \tag{2.3}$$

Contracting the above equation gives

$$\mathrm{Ric}(X, \nu) = -(n-1)\alpha^2\nu^\flat(X), \tag{2.4}$$

which yields $Q\nu = -(n-1)\alpha^2\nu$, where Q is Ricci operator defined by $g(QX, Y) = \mathrm{Ric}(X, Y)$. Differentiating $Q\nu = -(n-1)\alpha^2\nu$ along X implies that

$$(\nabla_X Q)\nu = -(n-1)\alpha^3 X - \alpha QX. \tag{2.5}$$

Taking the g-trace of the above equation gives (2.2). □

Let (M, g) be a Riemannian manifold. If there exists $\rho \in C^\infty(M)$, called the potential function, such that

$$\pounds_V g = 2\rho g$$

then we say that the vector field V is a *conformal vector field* (see Yano [14]). Moreover, V is *homothetic* when ρ is constant, whereas *Killing* when $\rho = 0$. Now, we recall the following result from Yano [14].

Lemma 2.2. *A conformal vector field V on a Riemannian n-manifold (M^n, g) satisfies*

$$(\pounds_V \mathrm{Ric})(X, Y) = -(n-2)g(\nabla_X D\rho, Y) - (\Delta\rho)g(Y, X),$$
$$\pounds_V s = -2\rho s - 2(n-1)\Delta\rho,$$

where $\Delta = \mathrm{div}\, D$ is the Laplacian operator of g.

Lemma 2.3. *If the metric of Riemannian three-manifold M equipped with a concurrent-recurrent vector field is a Yamabe soliton, then the scalar curvature of M is harmonic and the Yamabe soliton is shrinking with $\lambda = -6\alpha^2$.*

P r o o f. First, we take Lie-derivative to $g(\nu,\nu) = 1$ along V, and employ the equations (1.2) and (1.3) to get

$$(\mathcal{L}_V \nu^\flat)\nu = -\nu^\flat(\mathcal{L}_V \nu) = (s - \lambda). \tag{2.6}$$

Now, taking $n = 3$ and $\rho = s - \lambda$ in Lemma 2.2, we find

$$(\mathcal{L}_V \operatorname{Ric})(X,Y) = -g(\nabla_X Ds, Y) - (\Delta s)g(X,Y), \tag{2.7}$$

$$\mathcal{L}_V s = -2s(s - \lambda) - 4\Delta s. \tag{2.8}$$

In Riemannian three-manifolds, the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= g(Z,Y)QX - g(Z,X)QY + g(Z,QY)X - g(Z,QX)Y \\ &\quad - \frac{s}{2}\{g(Z,Y)X - g(Z,X)Y\}. \end{aligned} \tag{2.9}$$

Taking $Z = \nu$ in the above equation and using (2.3), we easily deduce

$$\operatorname{Ric}(X,Y) = \left(\frac{s}{2} + \alpha^2\right) g(Y,X) - \left(3\alpha^2 + \frac{s}{2}\right) \nu^\flat(X)\nu^\flat(Y). \tag{2.10}$$

Lie-differentiating of (2.10) along V and employing the equations (2.8) and (1.2), we find

$$\begin{aligned} (\mathcal{L}_V \operatorname{Ric})(X,Y) &= (2\alpha^2(s-\lambda) - 2\Delta s)g(X,Y) + (s(s-\lambda) + 2\Delta s)\nu^\flat(X)\nu^\flat(Y) \\ &\quad - \left(\frac{s}{2} + 3\alpha^2\right)\{(\mathcal{L}_V \nu^\flat)(X)\nu^\flat(Y) + \nu^\flat(X)(\mathcal{L}_V \nu^\flat)(Y)\}. \end{aligned}$$

Comparing the above equation with (2.7), we obtain

$$\begin{aligned} g(\nabla_X Ds, Y) &= (\Delta s - 2\alpha^2(s-\lambda))g(X,Y) - (s(s-\lambda) + 2\Delta s)\nu^\flat(X)\nu^\flat(Y) \\ &\quad + \left(\frac{s}{2} + 3\alpha^2\right)\{(\mathcal{L}_V \nu^\flat)(X)\nu^\flat(Y) + \nu^\flat(X)(\mathcal{L}_V \nu^\flat)(Y)\}. \end{aligned} \tag{2.11}$$

Replacing X and Y in the above equation by ν and utilizing (2.6), we see that

$$\nu(\nu s) = -\Delta s + 4\alpha^2(s - \lambda).$$

Now, we use (2.2) in the above equation in order to obtain

$$\Delta s = -4\alpha^2(\lambda + 6\alpha^2). \tag{2.12}$$

We replace Y by ν in (2.11) and use the equations (2.6) and (2.12) to deduce

$$g(\nabla_X Ds, \nu) = (4\alpha^2(\lambda + 6\alpha^2) + ((\alpha^2 - \frac{s}{2})s - \lambda))\nu^\flat(X) + (\frac{s}{2} + 3\alpha^2)(\mathcal{L}_V \nu^\flat)X.$$

On the other hand, differentiating (2.2) along X and utilizing (1.3) we obtain

$$g(\nabla_X Ds, \nu) = -3\alpha(Xs) - 2\alpha^2(s + 6\alpha^2)\nu^\flat(X).$$

Using the preceding equation in (2.11), we get

$$\begin{aligned} \left(\frac{s}{2} + 3\alpha^2\right)(\mathcal{L}_V \nu^\flat)(X) &= \{(s-\lambda)(\frac{s}{2} - \alpha^2) - 2\alpha^2(s + 6\alpha^2) \\ &\quad - 4\alpha^2(\lambda + 6\alpha^2)\}\nu^\flat(X) - 3\alpha(Xs). \end{aligned}$$

Substituting the above equation in (2.11) and using (2.12), we find

$$\begin{aligned} g(\nabla_X Ds, Y) &= -2\alpha^2(\lambda + s + 12\alpha^2)g(X,Y) + 2\alpha^2(\lambda - 3s - 12\alpha^2)\nu^\flat(X)\nu^\flat(Y) \\ &\quad - 3\alpha(Xs)\nu^\flat(Y) - 3\alpha(Ys)\nu^\flat(X), \end{aligned}$$

which further leads to

$$\begin{aligned}\nabla_X Ds = {}& -2\alpha^2(\lambda + s + 12\alpha^2)X + 2\alpha^2(\lambda - 3s - 12\alpha^2)\nu^\flat(X)\nu \\ & - 3\alpha(Xs)\nu - 3\alpha\nu^\flat(X)Ds. \end{aligned} \tag{2.13}$$

Replacing X in the previous equation by ν and applying (2.2), we get

$$\nabla_\nu Ds = -2\alpha^2(s + 6\alpha^2)\nu - 3\alpha Ds.$$

Operating the above equation by ∇_X gives us

$$\nabla_X \nabla_\nu Ds = -2\alpha^2(Xs)\nu - 3\alpha\nabla_X Ds - 2\alpha^2(s + 6\alpha^2)\nabla_X \nu. \tag{2.14}$$

On the other hand, differentiating (2.13) along ν, we get

$$\begin{aligned}\nabla_\nu \nabla_X Ds = {}& -2\alpha^2(\nu s)X - 2\alpha^2(\lambda + s + 12\alpha^2)\nabla_\nu X - 6\alpha^2(\nu s)\nu^\flat(X)\nu \\ & + 2\alpha^2(\lambda - 3s - 12\alpha^2)\nu^\flat(\nabla_\nu X)\nu - 3\alpha\nu(Xs)\nu \\ & - 3\alpha\nu^\flat(\nabla_\nu X)Ds - 3\alpha\nu^\flat(X)\nabla_\nu Ds. \end{aligned}$$

Again from (2.13) we immediately get

$$\begin{aligned}\nabla_{[X,\nu]} Ds = {}& -2\alpha^2(\lambda + s + 12\alpha^2)(\nabla_X \nu - \nabla_\nu X) - 3\alpha g(\nabla_X \nu - \nabla_\nu X, Ds)\nu \\ & + 2\alpha^2(\lambda - 3s - 12\alpha^2)\nu^\flat(\nabla_X \nu - \nabla_\nu X)\nu - 3\alpha\nu^\flat(\nabla_X \nu - \nabla_\nu X)Ds. \end{aligned} \tag{2.15}$$

Now, employing the equations (2.14)–(2.15), one can easily get

$$\begin{aligned}R(X,\nu)Ds = {}& -2\alpha^2(Xs)\nu - 3\alpha\nabla_X Ds - 2\alpha^2(s + 6\alpha^2)\nabla_X \nu + 2\alpha^2(\nu s)X \\ & + 2\alpha^2(\lambda + s + 12\alpha^2)\nabla_\nu X + 6\alpha^2(\nu s)\nu^\flat(X)\nu + 3\alpha\nu^\flat(X)\nabla_\nu Ds \\ & - 2\alpha^2(\lambda - 3s - 12\alpha^2)\nu^\flat(\nabla_\nu X)\nu + 3\alpha\nu(Xs)\nu + 3\alpha\nu^\flat(\nabla_\nu X)Ds \\ & + 2\alpha^2(\lambda + s + 12\alpha^2)(\nabla_X \nu - \nabla_\nu X) + 3\alpha g(\nabla_X \nu - \nabla_\nu X, Ds)\nu \\ & + 3\alpha\nu^\flat(\nabla_X \nu - \nabla_\nu X)Ds - 2\alpha^2(\lambda - 3s - 12\alpha^2)\nu^\flat(\nabla_X \nu - \nabla_\nu X)\nu. \end{aligned}$$

Contracting the above equation, we find

$$\mathrm{Ric}(\nu, Ds) = 10\alpha^2(\nu s) - 3\alpha\Delta s - 4\alpha^3(s + 6\alpha^2) + 6\alpha\nu(\nu s) + 4\alpha^3(\lambda + s + 12\alpha^2).$$

Making use of (2.2) together with (2.12) in the preceding equation, we obtain

$$\mathrm{Ric}(\nu, Ds) = 4\alpha^3(3\lambda + s + 30\alpha^2).$$

Now, we employ (2.4) and (2.2) in the above equation to deduce the value of soliton constant $\lambda = -6\alpha^2$, which means Yamabe soliton is expanding. Using this in (2.12), we have $\Delta s = 0$, that is, the scalar curvature is harmonic. □

3. Proof of the main results

Proof of Theorem 1.1. First, differentiate (1.2) along Z to achieve

$$(\nabla_Z \mathcal{L}_V g) = 2(Zs)g(X,Y). \tag{3.1}$$

In [14], Yano reveals the following relation:

$$\begin{aligned}(\mathcal{L}_V \nabla_X g - {}& \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y,Z) \\ & = -g((\mathcal{L}_V \nabla)(X,Z),Y) - g((\mathcal{L}_V \nabla)(X,Y),Z). \end{aligned}$$

Due to $\nabla g = 0$, it appears from the preceding equation that

$$(\nabla_X \mathcal{L}_V g)(X, Y) = g((\mathcal{L}_V \nabla)(X, Z), Y) + g((\mathcal{L}_V \nabla)(X, Y), Z).$$

Utilizing the symmetric property of $\mathcal{L}_V \nabla$, the foregoing equation brings into view

$$2g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Y \mathcal{L}_V g)(Z, X) + (\nabla_X \mathcal{L}_V g)(Y, Z) - (\nabla_Z \mathcal{L}_V g)(X, Y).$$

Utilizing (3.1) in the previous equation, we find

$$(\mathcal{L}_V \nabla)(X, Y) = (Xs)Y + (Ys)X - g(Y, X)Ds. \tag{3.2}$$

Taking differentiation of (3.2) covariantly along Z yields

$$(\nabla_Z \mathcal{L} \nabla)(X, Y) = g(\nabla_Z Ds, X)Y + g(\nabla_Z Ds, Y)X - g(Y, X)\nabla_Z Ds.$$

Applying the above relation on the following well known formula

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z),$$

appears that

$$(\mathcal{L}_V R)(X, Y)Z = g(\nabla_X Ds, Z)Y - g(Z, Y)\nabla_X Ds - g(\nabla_Y Ds, Z)X + g(X, Z)\nabla_Y Ds.$$

Replacing Z in the previous equation by ν and calling back $\lambda = -6\alpha^2$ and the equation (2.13), we infer

$$\begin{aligned}(\mathcal{L}_V R)(X, Y)\nu =& (4\alpha^2(s + 6\alpha^2)\nu^\flat(Y) + 3\alpha(Ys))X - 3\alpha(Ys)\nu^\flat(X)\nu \\ &- (4\alpha^2(s + 6\alpha^2)\nu^\flat(X) + 3\alpha(Xs))Y + 3\alpha(Xs)\nu^\flat(Y)\nu.\end{aligned} \tag{3.3}$$

At this point, we differentiate (2.3) and apply (1.2) to ensure

$$\begin{aligned}(\mathcal{L}_V R)(X, Y)\nu =& -\alpha^2\{(g(Y, \mathcal{L}_V \nu) + 2(s + 6\alpha^2)\nu^\flat(Y))X \\ &- (g(X, \mathcal{L}_V \nu) + 2(s + 6\alpha^2)\nu^\flat(X))Y\} - R(X, Y)\mathcal{L}_V \nu.\end{aligned} \tag{3.4}$$

Subtracting (3.3) from (3.4), we see that

$$\begin{aligned}R(X, Y)\mathcal{L}_V \nu = & (6\alpha^2(s + 6\alpha^2)\nu^\flat(X) + 3\alpha(Xs) + \alpha^2 g(\mathcal{L}_V \nu, X))Y + 3\alpha(Ys)\nu^\flat(X)\nu \\ &-(6\alpha^2(s + 6\alpha^2)\nu^\flat(Y) + 3\alpha(Ys) + \alpha^2 g(Y, \mathcal{L}_V \nu))X - 3\alpha(Xs)\nu^\flat(Y)\nu.\end{aligned}$$

Contracting the above equation, we may obtain that

$$\mathrm{Ric}(Y, \mathcal{L}_V \nu) = -6\alpha^2(s + 6\alpha^2) - 3\alpha(Ys) - 2\alpha^2 g(Y, \mathcal{L}_V \nu).$$

By the support of (2.10) and (2.6), the above equation shows that

$$(s + 6\alpha^2)\mathcal{L}_V \nu = -(s + 18\alpha^2)(s + 6\alpha^2)\nu - 6\alpha Ds. \tag{3.5}$$

If possible, we suppose that on an open subset $\mathcal{O}$ of M there holds $s \neq -6\alpha^2$. Then it appears from equation (3.5) that

$$\mathcal{L}_V \nu = -(s + 18\alpha^2)\nu - \frac{6\alpha}{s + 6\alpha^2} Ds. \tag{3.6}$$

Replacing Y with ν in the well-known formula (see [14]):

$$(\mathcal{L}_V \nabla)(X, Y) = \mathcal{L}_V \nabla_X Y - \nabla_X \mathcal{L}_V Y - \nabla_{[V,X]} Y,$$

and then utilizing $\lambda = -6\alpha^2$, (1.3) and (3.6) we obtain the following equality

$$\begin{aligned}(\mathcal{L}_V \nabla)(X, \nu) =& -\alpha(s + 30\alpha^2)\nu^\flat(X)\nu + \frac{s - 6\alpha^2}{s + 6\alpha^2}(Xs)\nu - \frac{12\alpha^2}{s + 6\alpha^2}\nu^\flat(X)Ds \\ &+ \alpha(s + 6\alpha^2)X - \frac{6\alpha}{(s + 6\alpha^2)^2}(Xs)Ds.\end{aligned} \tag{3.7}$$

On the other hand, replacing Y in (3.2) by ν and using (2.2) we have

$$(\mathcal{L}_V\nabla)(X,\nu) = (Xs)\nu - 2\alpha(s+6\alpha^2)X - \nu^\flat(X)Ds. \tag{3.8}$$

Comparing (3.7) with (3.8) implies that

$$\left(1 - \frac{s-6\alpha^2}{s+6\alpha^2}\right)(Xs)\nu - 3\alpha(s+6\alpha^2)X + \left(\frac{12\alpha^2}{s+6\alpha^2}\right)\nu^\flat(X)Ds$$
$$+\alpha(s+30\alpha^2)\nu^\flat(X)\nu + \frac{6\alpha}{(s+6\alpha^2)^2}(Xs)Ds = 0. \tag{3.9}$$

Finally, replacing X by ν in (3.9) and applying (2.2) we get

$$Ds = -2\alpha(s+6\alpha^2)\nu. \tag{3.10}$$

Using the previous equation in (3.6) gives us $\mathcal{L}_V\nu = -(s+6\alpha^2)\nu$. This relation together with (1.2) gives

$$(\mathcal{L}_V\nu^\flat)(X) = (\mathcal{L}_Vg)(X,\nu) + g(X,\mathcal{L}_V\nu) = (s+6\alpha^2)\nu^\flat(X). \tag{3.11}$$

With the help of Lemma 2.3, one can easily derived from (2.8) that

$$\mathcal{L}_Vs = -2s(s+6\alpha^2). \tag{3.12}$$

Now we write (3.10) as: $ds = -2\alpha(s+6\alpha^2)\nu^\flat$. Now, we take Lie derivative to this equation in order to deduce

$$\mathcal{L}_Vds = -2\alpha(\mathcal{L}_Vs)\nu^\flat - 2\alpha(s+6\alpha^2)\mathcal{L}_V\nu^\flat.$$

From (3.12), the preceding equation transforms into

$$\mathcal{L}_Vds = 4\alpha s(s+6\alpha^2)\nu^\flat - 2\alpha(s+6\alpha^2)\mathcal{L}_V\nu^\flat. \tag{3.13}$$

Operating the equation (3.12) by d and using $ds = -2\alpha(s+6\alpha^2)\nu^\flat$, we have

$$\mathcal{L}_Vds = 4\alpha(s+6\alpha^2)^2\nu^\flat + 4\alpha s(s+6\alpha^2)\nu^\flat, \tag{3.14}$$

where used the fact that Lie-derivative commutes with the exterior derivative. Now, comparing (3.13) with (3.14), leads us to the following formula

$$(s+6\alpha^2)\{\mathcal{L}_V\nu^\flat + 2(s+6\alpha^2)\}\nu^\flat = 0.$$

As we know $s \neq -6\alpha^2$ on $\mathcal{O}$, we must have

$$\mathcal{L}_V\nu^\flat = -2(s+6\alpha^2)\nu^\flat.$$

Comparing the previous equation with (3.11) shows that the scalar curvature $s = -6\alpha^2$ on $\mathcal{O}$ and this is a contradiction. So, we must have $s = -6\alpha^2$ on M. Substituting this in (2.10) we see that $\mathrm{Ric} = (1-n)\alpha^2 g$ which along with (2.9) shows that the manifold is of constant negative curvature $-\alpha^2$, and consequently, V is Killing. This concludes the proof of the theorem.

Proof of Theorem 1.2. We may write the equation (1.4) as

$$\nabla_X Df = \frac{1}{\tau}g(X,Df)Df + (s-\lambda)Df. \tag{3.15}$$

Using the previous equation in the definition of curvature tensor, we find

$$R(X,Y)Df = \frac{(s-\lambda)}{\tau}\{(Yf)X - (Xf)Y\} + (Xs)Y - (Ys)X. \tag{3.16}$$

Replacing X by ν in the previous equation and comparing the obtained equation with $R(\nu,X)Df = -\alpha^2\{g(X,Df)\nu - (\nu f)X\}$ (which follows from (2.3)), we have

$$-\alpha^2(Yf)\nu + \alpha^2(\nu f)Y = \frac{s-\lambda}{\tau}(Yf)\nu - \frac{s-\lambda}{\tau}(\nu f)Y + (\nu s)Y - (Ys)\nu. \tag{3.17}$$

On the other hand, we contract the equation (3.16) over X to find the expression of Ricci tensor as

$$\mathrm{Ric}(Y, Df) = \frac{(n-1)(s-\lambda)}{\tau}(Yf) - (n-1)(Ys). \tag{3.18}$$

Replacing Y in the above equation by ν and utilizing (2.4) imply

$$(\nu s) - \alpha^2(\nu f) = \frac{s-\lambda}{\tau}(\nu f), \tag{3.19}$$

which is a relation involving the scalar curvature and the potential function of the τ-quasi Yamabe gradient soliton. Feeding (3.19) in (3.17), we obtain

$$\tau Ds = (s - \lambda + \tau\alpha^2)Df. \tag{3.20}$$

Differentiating (3.20) with respect to X and using (3.15), we reach at

$$\tau\nabla_X Ds = (Xs)Df + (Xf)Df + (s - \lambda + \tau\alpha^2)(s-\lambda)X. \tag{3.21}$$

Taking scalar product of (3.16) with Df, we have $(Xs)Df = (Xf)Ds$. Employing this in (3.21), we find

$$\tau\nabla_X Ds = 2g(X, Df)Ds + (s - \lambda + \tau\alpha^2)(s-\lambda)X. \tag{3.22}$$

At this stage, we use (3.20) in (3.18) in order to ensure

$$QDf = -\alpha^2(n-1)Df.$$

Differentiating the previous equation with respect to X and calling back (3.15) give

$$(\nabla_X Q)Df + (s-\lambda)QX + \alpha^2(n-1)(s-\lambda)X = 0. \tag{3.23}$$

On the other hand, from second Bianchi identity one can find

$$\mathrm{trace}_g\{X \to (\nabla_X Q)Y\} = (\mathrm{div}\ Q)(Y) = \frac{1}{2}Y(s).$$

Now, we contract (3.23) over X and utilize the above identity to deduce

$$g(Ds, Df) + 2(s-\lambda)(s + n\alpha^2(n-2)) = 0. \tag{3.24}$$

Combining the equations (3.15), (3.20) and (3.22) one can easily find

$$\begin{aligned}\frac{3}{\tau}g(X, Df)g(Ds, Df) &+ 2\frac{(s-\lambda+\tau\alpha^2)(s-\lambda)}{\tau}g(X, Df)\\ &= -2\frac{(2s-2\lambda+\tau\alpha^2)(s-\lambda+\tau\alpha^2)}{\tau}g(X, Df) = 0.\end{aligned}$$

Setting $X = Df$ in the above equation and using the fact that $|Df| \neq 0$ (as the soliton is non-trivial), we have

$$\frac{3}{\tau}g(Ds, Df) + 2\frac{(s-\lambda+\tau\alpha^2)(s-\lambda)}{\tau} + 2\frac{(2s-2\lambda+\tau\alpha^2)(s-\lambda+\tau\alpha^2)}{\tau} = 0.$$

Now, we employ (3.24) in the preceding equation to deduce

$$(s-\lambda+\tau\alpha^2)(3s-3\lambda+\tau\alpha^2) - 3(s-\lambda)(s+n\alpha^2(n-1)) = 0.$$

The above equation shows that s is constant. So that we have $\nu(s) = 0$, which together tracing of (2.5) gives that $s = -n\alpha^2(n-1)$. This completes the proof.

4. Example

In this section, we construct a Riemannian manifold (M, g) admitting a concurrent-recurrent vector field for which the metric g is a Yamabe soliton.

Consider a manifold $M = \{(u, v, w) \in \mathbb{R}^3 : v > 0, w \neq 0\}$ with a coordinate system (u, v, w). Let us define a Riemannian metric on M as

$$g = 2\alpha^2 w^2 (\mathrm{d}u)^2 + \frac{\alpha^2 w^2}{2v}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + \frac{\alpha^2 w^2}{4v^2}(\mathrm{d}v)^2 + \frac{1}{\alpha^2 w^2}(\mathrm{d}w)^2,$$

where $\alpha(\neq 0) \in \mathbb{R}$. From Koszul's formula, one can easily compute

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = -2\alpha^4 w^3 \frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} = -\frac{\alpha^4 w^3}{2v}\frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial w} = \frac{1}{w}\frac{\partial}{\partial u},\\
&\nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial u} = -\frac{\alpha^4 w^3}{2v}\frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v} = -\frac{1}{v}\frac{\partial}{\partial v} - \frac{\alpha^4 w^3}{4v^2}\frac{\partial}{\partial w}, &&\nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial w} = \frac{1}{w}\frac{\partial}{\partial v},\\
&\nabla_{\frac{\partial}{\partial w}} \frac{\partial}{\partial u} = \frac{1}{w}\frac{\partial}{\partial u}, &&\nabla_{\frac{\partial}{\partial w}} \frac{\partial}{\partial v} = \frac{1}{w}\frac{\partial}{\partial v}, &&\nabla_{\frac{\partial}{\partial w}} \frac{\partial}{\partial w} = -\frac{1}{w}\frac{\partial}{\partial w}.
\end{aligned}$$

Let us take $\nu = \alpha w \frac{\partial}{\partial w}$. Then, from above we can verify that

$$\nabla_{X_i}\nu = \alpha\{X_i - \nu^\flat(X_i)\nu\},$$

for all $1 \leq i \leq 3$, where $X_1 = \frac{\partial}{\partial u}$, $X_2 = \frac{\partial}{\partial v}$ and $X_3 = \frac{\partial}{\partial w}$. Thus, the vector field $\nu = \alpha w \frac{\partial}{\partial w}$ is concurrent-recurrent vector field. Now we use Levi-Civita connection to find the non-zero components of curvature tensor as given below

$$\begin{aligned}
&R(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})\frac{\partial}{\partial u} = -\frac{\alpha^4 w^2}{2v}\frac{\partial}{\partial u} + 2\alpha^4 w^2 \frac{\partial}{\partial v}, &&R(\frac{\partial}{\partial v}, \frac{\partial}{\partial w})\frac{\partial}{\partial v} = \frac{\alpha^4 w^2}{4v^2}\frac{\partial}{\partial w},\\
&R(\frac{\partial}{\partial u}, \frac{\partial}{\partial w})\frac{\partial}{\partial w} = -\frac{1}{w^2}\frac{\partial}{\partial u}, &&R(\frac{\partial}{\partial u}, \frac{\partial}{\partial w})\frac{\partial}{\partial v} = \frac{\alpha^4 w^2}{2v}\frac{\partial}{\partial w},\\
&R(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})\frac{\partial}{\partial v} = \frac{\alpha^4 w^2}{2v}\frac{\partial}{\partial v} - \frac{\alpha^4 w^2}{4v^2}\frac{\partial}{\partial u}, &&R(\frac{\partial}{\partial v}, \frac{\partial}{\partial w})\frac{\partial}{\partial w} = -\frac{1}{w^2}\frac{\partial}{\partial v},\\
&R(\frac{\partial}{\partial u}, \frac{\partial}{\partial w})\frac{\partial}{\partial u} = 2\alpha^4 w^2 \frac{\partial}{\partial w}, &&R(\frac{\partial}{\partial v}, \frac{\partial}{\partial w})\frac{\partial}{\partial u} = \frac{\alpha^4 w^2}{2v}\frac{\partial}{\partial w},
\end{aligned}$$

From the curvature tensor we find the scalar curvature as $s = -6\alpha^2$. Also, it is not hard to verify that

$$R(X_i, X_j)X_k = -\alpha^2(g(X_j, X_k)X_i - g(X_i, X_k)X_j),$$

for all $1 \leq i, j, k \leq 3$. This shows that (M, g) is of constant curvature $-\alpha^2$.

Now we shall show that the metric g is a Yamabe soliton on M. Let

$$V = \frac{ln(v)}{2}\frac{\partial}{\partial u} - 2v(ln(v) + 2u)\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}$$

be a vector field on M. It is not hard to see that $\mathcal{L}_V g = 0$, and so V is Killing. Thus, we see that

$$\mathcal{L}_V g = (s - \lambda)g$$

for $\lambda = -6\alpha^2$. Hence, g is a Yamabe soliton having the potential vector field $V = \frac{ln(v)}{2}\frac{\partial}{\partial u} - 2v(ln(v) + 2u)\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}$ and the soliton constant $\lambda = -6\alpha^2$. Also, we see that this example verifies our Theorem 1.1.

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