Yet more versions of $d = 11$ supergravity

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Abstract. Extending previous results, we construct two further versions of $d = 11$ supergravity. The transformation laws are given in a form where the SO(1, 10) local Lorentz group is replaced by SO(1, 4) × USp(8) and SO(1, 5) × [SO(5) × SO(5)], respectively. The spinless fields are assigned to the corresponding representations of $E_{6(+6)}$ and $E_{5(+5)}$.

1. Introduction

It has been known for some time that maximally extended supergravities in D dimensions obtained from $d = 11$ supergravity [1] are manifestly invariant under an internal symmetry group SL$(11 - D)_{\text{global}}$ × SO$(11 - D)_{\text{local}}$ [2, 3]. This symmetry can be enlarged to the product of a non-compact global group G_{11-D} and a compact local group H_{11-D}. Assuming that the scalar sector in these theories is governed by a non-linear σ model based on the coset G_{11-D}/H_{11-D}, one can deduce the relevant groups by equating the number of scalars obtained in the dimensional reduction with the dimension of this coset space. In this way, one gets [2, 3] the results presented in table 1†.

In [4] it was shown that some of these hidden symmetries are actually present in the $d = 11$ supergravity itself: there exists a reformulation where the tangent space group SO(1, 3) × SO(7) is replaced by SO(1, 3) × SU(8) and the spinless degrees of freedom are assigned to a representation of $E_{7(+7)}$. A crucial step in this construction

Table 1.

D	G_{11-D}	H_{11-D}
8	$E_{3(+3)} = \text{SL}(3, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$	SO(3) × SO(2)
7	$E_{4(+4)} = \text{SL}(5, \mathbb{R})$	SO(5)
6	$E_{5(+5)} = \text{SO}(5, 5)$	SO(5) × SO(5)
5	$E_{6(+6)}$	USp(8)
4	$E_{7(+7)}$	SU(8)
3	$E_{8(+8)}$	SO(16)

† For $D \geq 9$ the field A_{MNP} does not give any scalars in the reduction. Hence one retains only GL$(2, \mathbb{R})_{\text{global}}$ × SO$(2)_{\text{local}}$ after the Weyl rescaling for $D = 9$.

was the introduction of 'composite' gauge connections in the internal dimensions so as to covariantise the theory with respect to SU(8) transformations depending on the internal coordinates. Subsequently, the corresponding version of $d = 11$ supergravity with tangent space group $SO(1, 2) \times SO(16)$ was constructed in [5]. These results suggest that, in fact, all of the hidden symmetries displayed in table 1 can be realised in the full $d = 11$ theory. In this paper we show that this is indeed the case by presenting explicit results for $D = 5$ and $D = 6$ (we have not considered $D \geq 7$ since these cases are less interesting and unlikely to furnish any new insights). This confirms the conclusion that $d = 11$ supergravity is a multifaceted theory whose ultimate formulation (and possible embedding into an ancestor dual model) remains to be found. It would, of course, be even more interesting if one could extend the results now established for $3 \leq D \leq 6$ to $D \leq 2$ where one expects the emergence of infinite-dimensional groups†.

We now briefly summarise our notation and conventions; more details can be found in [4, 5]. The only difference with [4, 5] is that we use a metric with signature $(+, -, \ldots, -)$. The Lagrangian of $d = 11$ supergravity is given by η

$$\mathcal{L} = -\tfrac{1}{2}ER - \tfrac{1}{2}iE\bar{\Psi}_M\tilde{\Gamma}^{MNP}D_N\Psi_P - \tfrac{1}{48}EF^2_{MNPQ}$$
$$+ (1/\sqrt{2})[1/(12)^3]\varepsilon^{M\ldots X}F_{MNPQ}F_{RSTU}A_{VWX}$$
$$+ (\sqrt{2}/192)EF_{MNPQ}(\bar{\Psi}_R\tilde{\Gamma}^{MNPQRS}\Psi_S + 12\bar{\Psi}^M\tilde{\Gamma}^{NP}\Psi^Q) \tag{1.1}$$

where, of course,

$$\{\tilde{\Gamma}^A, \tilde{\Gamma}^B\} = 2\eta^{AB} \qquad \tilde{\Gamma}^{A_1\ldots A_{11}} = -i\varepsilon^{A_1\ldots A_{11}}. \tag{1.2}$$

The transformation laws are

$$\delta E_M^A = -\tfrac{1}{2}i\bar{\varepsilon}\tilde{\Gamma}^A\Psi_M$$
$$\delta A_{MNP} = \tfrac{1}{8}\sqrt{2}\bar{\varepsilon}\tilde{\Gamma}_{[MN}\Psi_{P]} \tag{1.3}$$
$$\delta\Psi_M = D_M\varepsilon + \tfrac{1}{288}i\sqrt{2}(\tilde{\Gamma}_M{}^{PQRS} - 8\delta_M^P\tilde{\Gamma}^{QRS})\varepsilon F_{PQRS}.$$

Higher-order fermionic terms are neglected throughout.

2. Field redefinitions and transformation laws

The starting point is a $D + (11 - D)$ coordinate split. The curved and flat $d = 11$ indices M and A are decomposed into (μ, m) and (α, a), respectively. Fixing the gauge

$$E_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix} \tag{2.1}$$

reduces the local $SO(1, 10)$ Lorentz group to $SO(1, D-1) \times SO(11-D)$. The Weyl rescaling

$$e'_\mu{}^\alpha = \Delta^s e_\mu{}^\alpha \qquad \Delta \equiv \det e_m{}^a \qquad s \equiv 1/(D-2) \tag{2.2}$$

leads to the standard Einstein action in D dimensions (for $D > 2$). The fermionic fields are redefined as follows:

$$\Psi'_\mu = \Delta^{-s/2}e'_\mu{}^\alpha(\Psi_\alpha + s\tilde{\Gamma}_\alpha\tilde{\Gamma}^a\Psi_a) = e'_\mu{}^\alpha\Psi'_\alpha$$
$$\Psi'_a = \Delta^{-s/2}\Psi_a \tag{2.3}$$
$$\varepsilon' = \Delta^{s/2}\varepsilon.$$

† For some recent results concerning maximally extended supergravity in two dimensions, see [6].

As a general rule, quantities with index μ are obtained from the flat ones via $e_\mu{}^\alpha$ and without the $E_\mu{}^a$ contribution (so that we have a scalar under $(11-D)$-dimensional general coordinate transformations $\xi^m(x^\mu, y^m)$). In particular, we use [4, 5]

$$\mathcal{D}_\mu = e_\mu{}^\alpha \partial_\alpha = \partial_\mu - B_\mu{}^m \partial_m$$

$$\omega_{\mu AB} = \tfrac{1}{2} e_\mu{}^\alpha (\Omega_{AB\alpha} - \Omega_{B\alpha A} - \Omega_{\alpha AB})$$

(2.4)

where

$$\Omega_{AB}{}^C = 2 E_{[A}{}^M E_{B]}{}^N \partial_M E_N{}^C.$$

The supersymmetry transformation of the spin-2 field is given by

$$\delta e'_\mu{}^\alpha = \delta_{\text{susy}}(\Delta^s e_\mu{}^\alpha) + E_\mu{}^A \Omega_A{}^\alpha$$

$$= -\tfrac{1}{2} i \bar{\varepsilon}' \tilde{\Gamma}^\alpha \Psi'_\mu.$$

(2.5)

The compensating $SO(1, 10)$ rotation

$$\Omega_{\alpha a} = \tfrac{1}{2} i \bar{\varepsilon} \tilde{\Gamma}_\alpha \Psi_a \qquad \Omega_{\alpha \beta} = \tfrac{1}{2} i s \bar{\varepsilon} \tilde{\Gamma}_{\alpha\beta} \tilde{\Gamma}^a \Psi_a$$

(2.6)

restores the gauge (2.1). The spin-1 fields transform into

$$\delta B_\mu{}^m = -\tfrac{1}{2} i \Delta^{-s} e_a{}^m [\bar{\varepsilon}' \tilde{\Gamma}^a \Psi'_\mu + \bar{\varepsilon}' \tilde{\Gamma}'^\mu (\eta^{ab} + s \tilde{\Gamma}^a \tilde{\Gamma}^b) \Psi'_b].$$

(2.7)

Variation of the spin-$\tfrac{3}{2}$ fields Ψ'_μ (2.3) yields the lengthy expression

$$\delta \Psi'_\mu = [\mathcal{D}_\mu - \tfrac{1}{4}(\hat{\omega}_{\mu\alpha\beta} \tilde{\Gamma}^{\alpha\beta} + \omega'_{\mu ab} \tilde{\Gamma}^{ab}) - \tfrac{1}{2} s \partial_m B_\mu{}^m] \varepsilon'$$
$$+ (\tfrac{1}{5} s \tilde{\Gamma}'_\mu \tilde{\Gamma}^{\alpha\beta} - e'^{[\alpha}_\mu \tilde{\Gamma}^{\beta]})(-\tfrac{1}{4} \Delta^s \Omega'_{\alpha\beta a} \tilde{\Gamma}^a + \tfrac{1}{8}\sqrt{2} \Delta^{-s} F_{\alpha\beta cd} \tilde{\Gamma}^{cd}) \varepsilon'$$
$$- s \Delta^{-s} e_a{}^m \tilde{\Gamma}^a [\partial_m + \tfrac{1}{4} \Omega'_{m\alpha\beta} \tilde{\Gamma}^{\alpha\beta} - \tfrac{1}{4} \omega_{mab} \tilde{\Gamma}^{ab} - \tfrac{1}{2}(1+s)(\Delta^{-1} \partial_m \Delta)] \tilde{\Gamma}'_\mu \varepsilon'$$
$$+ (\tfrac{1}{2} - s) \Omega'_{a(\alpha\beta)} e'^\alpha_\mu \tilde{\Gamma}^\beta \tilde{\Gamma}^a \varepsilon' + \tfrac{1}{96}\sqrt{2} \Delta^{-s} e'^\alpha_\mu [-s F_{abcd} \tilde{\Gamma}_\alpha \tilde{\Gamma}^{abcd} - 4 F_{abcd} \tilde{\Gamma}^{bcd}$$
$$+ (8s \tilde{\Gamma}_\alpha \tilde{\Gamma}^{\beta\gamma\delta} - 12 \delta_\alpha^{[\beta} \tilde{\Gamma}^{\gamma\delta]}) F_{\beta\gamma\delta d} \tilde{\Gamma}^d + (3s \tilde{\Gamma}_\alpha \tilde{\Gamma}^{\beta\gamma\delta\varepsilon} - 4 \delta_\alpha^{[\beta} \tilde{\Gamma}^{\gamma\delta\varepsilon]}) F_{\beta\gamma\delta\varepsilon}] \varepsilon' \quad (2.8)$$
$$\hat{\omega}'_{\mu\alpha\beta} \equiv \omega'_{\mu\alpha\beta} + 2 s e'_{\mu[\alpha} e'^\nu_{\beta]} \partial_m B_\nu{}^m.$$

(2.9)

3. Covariantisation for $D = 5$

To establish local $USp(8)$ covariance we give some notation and definitions for the $5+6$ coordinate split. The $d = 11$ γ matrices are represented by

$$\tilde{\Gamma}^\alpha = \gamma^\alpha \otimes \Gamma^* \qquad \Gamma^* = i\Gamma^{123456} = i\Omega = i\Gamma^7$$

$$\tilde{\Gamma}^a = 1 \otimes \Gamma^a.$$

(3.1)

For γ^α $(\alpha = 0, \ldots, 4)$ we use the conventions of [3]. Γ^a and Ω are real antisymmetric 8×8 matrices:

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} 1 \qquad \Omega^2 = -1.$$

(3.2)

Accordingly, the $D = 11$ Dirac indices are split as $(\underline{\alpha}, A)$ where $\underline{\alpha}$ is the $D = 5$ index and A is the 'internal' $SO(6)$ spinor index. We will suppress the index $\underline{\alpha}$ in the following. The $SO(6)$ part of the local Lorentz group acting on Majorana spinors can be extended to $USp(8)$ with the 36 generators $\Omega, \Gamma^{ab}, i\Gamma^{abc}$ satisfying

$$\Lambda^+ = -\Lambda$$

$$(\Omega\Lambda)^{\text{T}} = \Omega\Lambda.$$

(3.3)

We can raise and lower USp(8) indices by means of the symplectic metric Ω. Thus

$$\Psi_A = \Omega_{AB}\Psi^B \tag{3.4}$$

transforms in the complex conjugate representation of Ψ^B by virtue of (3.3). It follows that USp(8) is compatible with the $d = 11$ Majorana condition which in turn is equivalent to the $D = 5$ pseudo-Majorana condition [3]

$$\Psi^A = \gamma_5\Omega_{AB}(\Psi^B)^*. \tag{3.5}$$

The next step is to regroup the 48 spin-$\frac{1}{2}$ fields $\Psi_a{}^A$ into a representation of USp(8). This is done as follows [2, 3]:

$$\chi_{ABC} \equiv \tfrac{3}{2}\sqrt{2}\mathrm{i}\Gamma^a_{\langle AB}\Psi'_{C\rangle a}$$
$$= \tfrac{3}{2}\sqrt{2}\mathrm{i}\{\Gamma^a_{[AB}\Psi'_{C]a} + \tfrac{1}{3}\Omega_{[AB}\Gamma^a_{C]D}\Psi'^D_a\} \tag{3.6}$$

where $\langle\ldots\rangle$ denotes antisymmetric tensors which are traceless with respect to Ω. χ_{ABC} satisfies the reality condition

$$\chi^{ABC} = \gamma_5(\chi_{ABC})^*. \tag{3.7}$$

We note the identity

$$\bar\chi^{ABC}\gamma^\mu D_\mu\chi_{ABC} = 6\mathrm{i}\bar\Psi_a(\eta^{ab} + \tfrac{1}{3}\Gamma^a\Gamma^b)\tilde\Gamma^\mu D_\mu\Psi_b \tag{3.8}$$

which renders the spin-$\frac{1}{2}$ kinetic term diagonal.

From (2.5) and (2.7) we obtain the USp(8) covariant transformation laws

$$\delta e'^\alpha_\mu = \tfrac{1}{2}\bar\varepsilon^A\gamma^\alpha\Psi_{\mu A}$$
$$\delta B^m_\mu = -\tfrac{1}{2}e^m_{AB}(\bar\varepsilon^A\Psi^B_\mu - \tfrac{1}{4}\sqrt{2}\bar\varepsilon_C\gamma'_\mu\chi^{ABC}) \tag{3.9}$$

where the generalised vielbein e^m_{AB} is defined as

$$e^m_{AB} \equiv \mathrm{i}\Delta^{-1/3}e_a{}^m\Gamma^a_{AB} \tag{3.10}$$

and thereby assigned to the 27 representation of USp(8). As explained in [4] one must introduce a local USp(8) to avoid the emergence of new degrees of freedom; (3.10) then corresponds to a special gauge choice. Raising indices gives $e^{mAB} = (e^m_{AB})^*$.

Using properties of the $d = 7$ Clifford algebra [2, 4] the supersymmetry transformation of e^m_{AB} can be cast into the form

$$\delta_{\mathrm{susy}}e^m_{AB} = \tfrac{1}{2}\sqrt{2}\Sigma_{ABCD}e^{mCD} + 2\Lambda^C_{[A}e^m_{B]C} \tag{3.11}$$

with

$$\Sigma_{ABCD} = \bar\varepsilon_{[A}\chi_{BCD]} + \tfrac{1}{24}\varepsilon_{ABCDEFGH}\bar\varepsilon^E\chi^{FGH} \tag{3.12}$$

and the USp(8) parameter

$$\Lambda^A{}_B = -\tfrac{1}{8}\mathrm{i}(\bar\varepsilon\Gamma_a\Psi_b)\Gamma^{ab} + \tfrac{1}{16}\mathrm{i}(\bar\varepsilon\Gamma_{ab}\Psi_c)\Gamma^{abc}. \tag{3.13}$$

Σ_{ABCD} is antisymmetric, traceless and pseudoreal: $\Sigma^{ABCD} = (\Sigma_{ABCD})^*$ and thus transforms in the 42 representation of USp(8). Discarding the local USp(8) rotation we arrive at a manifestly covariant transformation law for e^m_{AB}. Furthermore the right-hand side of (3.11) can be interpreted as a (local) E_6 rotation in the fundamental 27 representation.

We must then rewrite the fermionic variations. Just as in [4, 5], (2.8) can be simplified by the introduction of further USp(8) covariant quantities. These are found in exactly the same fashion as in [4, 5] and are given by (we omit all primes in the following)

$$\mathcal{B}_\mu{}^A{}_B = -\tfrac{1}{2}\omega_{\mu\,ab}\Gamma^{ab}_{AB} - \tfrac{1}{12}\sqrt{2}i\Delta^{-1/3}e_\mu{}^\alpha F_{\alpha bcd}\Gamma^{bcd}_{AB} - \tfrac{1}{48}\sqrt{2}\Delta^{-1/3}e_\mu{}^\alpha\varepsilon_{\alpha\beta\gamma\delta\varepsilon}F^{\beta\gamma\delta\varepsilon}\Omega_{AB} \tag{3.14}$$

$$\mathcal{A}_{\mu\,ABCD} = \tfrac{3}{4}\Omega_{\mu\,(ab)}\Gamma^a_{\langle AB}\Gamma^b_{CD\rangle} - \tfrac{1}{8}\sqrt{2}i\Delta^{-1/3}e_\mu{}^\alpha F_{\alpha abc}\Gamma^a_{[AB}\Gamma^{bc}_{CD]}$$
$$- \tfrac{1}{192}\sqrt{2}\Delta^{-1/3}e_\mu{}^\alpha\varepsilon_{\alpha\beta\gamma\delta\varepsilon}F^{\beta\gamma\delta\varepsilon}\Gamma_{a[AB}\Gamma^{a7}_{CD]} \tag{3.15}$$

$$\mathcal{B}_m{}^A{}_B = \tfrac{1}{2}e_a{}^n\partial_m e_{nb}\Gamma^{ab}_{AB} - \tfrac{1}{48}\sqrt{2}ie_m{}^a F_{abcd}\Gamma^{bcd}_{AB} \tag{3.16}$$

$$\mathcal{A}_{m\,ABCD} = \tfrac{3}{4}e_a{}^n\partial_m e_{nb}\Gamma^a_{\langle AB}\Gamma^b_{CD\rangle} - \tfrac{1}{32}\sqrt{2}ie_m{}^a F_{abcd}\Gamma^b_{[AB}\Gamma^{cd}_{CD]}. \tag{3.17}$$

In addition we introduce the spin-1 field strength

$$\mathcal{G}_{\alpha\beta}{}^A{}_B = \tfrac{1}{4}i\Delta^{1/3}\Omega_{\alpha\beta a}\Gamma^{a7}_{AB} - \tfrac{1}{8}\sqrt{2}\Delta^{-1/3}F_{\alpha\beta cd}\Gamma^{cd7}_{AB} + \tfrac{1}{24}\sqrt{2}i\Delta^{-1/3}\varepsilon_{\alpha\beta cd}F^{\gamma\delta\varepsilon}{}_a\Gamma^a_{AB}. \tag{3.18}$$

The USp(8) covariant 'generalised vielbein postulate' is

$$\mathcal{D}_\mu e^m_{AB} + \tfrac{1}{3}\partial_n B^n_\mu e^m_{AB} + \partial_n B^m_\mu e^n_{AB} + \mathcal{B}_\mu{}^C{}_{[A}e^m_{B]C} + \mathcal{A}_{\mu\,ABCD}e^{m\,CD} = 0 \tag{3.19}$$

$$\partial_m e^n_{AB} + \mathcal{B}_m{}^C{}_{[A}e^n_{B]C} + \mathcal{A}_{m\,ABCD}e^{n\,CD} = 0. \tag{3.20}$$

With the help of the $SO(1,4) \times USp(8)$ covariant derivatives

$$\begin{aligned}D_\mu\varepsilon^A &\equiv \mathcal{D}_\mu\varepsilon^A - \tfrac{1}{4}\hat\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon^A + \tfrac{1}{2}\mathcal{B}_\mu{}^A{}_B\varepsilon^B\\ D_m\varepsilon^A &\equiv \partial_m\varepsilon^A + \tfrac{1}{4}e^\mu_\alpha\partial_m e_{\mu\beta}\gamma^{\alpha\beta}\varepsilon^A + \tfrac{1}{2}\mathcal{B}_m{}^A{}_B\varepsilon^B\end{aligned} \tag{3.21}$$

one finally obtains

$$\delta\chi_{ABC} = \tfrac{3}{2}\sqrt{2}e^m_{\langle AB}D_m\varepsilon_{C\rangle} - \tfrac{3}{4}\sqrt{2}\gamma^{\alpha\beta}\mathcal{G}_{\alpha\beta\langle AB}\varepsilon_{C\rangle} - \sqrt{2}\gamma^\mu\mathcal{A}_{\mu\,ABCD}\varepsilon^D$$
$$- \tfrac{3}{4}\sqrt{2}e^{m\,DE}\mathcal{A}_{m\,DE\langle AB}\varepsilon_{C\rangle} - \sqrt{2}\mathcal{A}_{m\,ABCD}e^{m\,DE}\varepsilon_E \tag{3.22}$$

$$\delta\Psi^A_\mu = [D_\mu(\hat\omega,\mathcal{B}_\mu) - \tfrac{1}{6}\partial_m B^m_\mu]\varepsilon^A + \tfrac{1}{6}(\gamma^{\alpha\beta}\gamma_\mu - 2e_\mu{}^\alpha\gamma^\beta)\mathcal{G}_{\alpha\beta}{}^A{}_B\varepsilon^B$$
$$+ \tfrac{1}{3}e^{mAB}[D_m(\gamma_\mu\varepsilon_B) + \tfrac{1}{4}(\partial_m g_{\mu\nu})\gamma^\nu\varepsilon_B] - \tfrac{1}{6}e^m_{CD}\mathcal{A}^{ABCD}_m\gamma_\mu\varepsilon_B. \tag{3.23}$$

In the reduction to $D = 5$ where the y dependence is dropped, these results agree with those of [3].

4. Covariantisation for $D = 6$

Since the procedure should be clear, we just quote the results for $D = 6$ in this section. According to the $6 + 5$ coordinate split the $d = 11$ γ matrices are represented by

$$\begin{aligned}\tilde\Gamma^\alpha &= \gamma^\alpha\otimes\mathbf{1} & \gamma^{\alpha\beta\gamma\delta\varepsilon\zeta} &= \varepsilon^{\alpha\beta\gamma\delta\varepsilon\zeta}\Gamma^*\\ \tilde\Gamma^a &= \Gamma^*\otimes\Gamma^a & \Gamma^{abcde} &= -i\varepsilon^{abcde}.\end{aligned} \tag{4.1}$$

γ^α and Γ^a correspond to Γ^α and γ^α of the last section with an additional factor of i due to the metric $\eta^{\alpha\beta} = \text{diag}(+, -, \ldots, -)$, $\eta^{ab} = \text{diag}(-, \ldots, -)$.

In $D = 6$ there exist Weyl spinors. The projectors are

$$\Pi_\pm = \tfrac{1}{2}(1 \pm \Gamma^*)\otimes\mathbf{1}. \tag{4.2}$$

We want to extend the internal SO(5) Lorentz group (table 1) so we assign the chiral spinor components (which are still (pseudo)-Majorana spinors) to representations of $SO(5) \times SO(5)$:

$$\varepsilon_A \equiv \Pi_+ \varepsilon' \qquad \Psi_{\mu A} \equiv \Pi_+ \Psi'_\mu$$

$$\varepsilon_{\bar{A}} \equiv \Pi_- \varepsilon' \qquad \Psi_{\mu \bar{A}} \equiv \Pi_- \Psi'_\mu. \tag{4.3}$$

A and $\bar{A}$ denote the spinor indices of the first and the second SO(5) in $SO(5) \times SO(5)$, respectively; a and $\bar{a}$ are the corresponding vector indices. The original SO(5) is, of course, embedded as the diagonal subgroup (just as for $D = 3$ [5]).

We redefine the spin-$\frac{1}{2}$ fields

$$\chi_{a\bar{A}} \equiv \tfrac{1}{2} \Pi_+ \Gamma^b \Gamma_a \Psi'_b$$

$$\chi_{\bar{a}A} \equiv \tfrac{1}{2} \Pi_- \Gamma^b \Gamma_a \Psi'_b \tag{4.4}$$

and introduce the '16-bein'

$$e^m{}_A{}^{\bar{B}} \equiv i\Delta^{-1/4} e_a{}^m \Gamma^a_{AB}. \tag{4.5}$$

This yields $SO(5) \times SO(5)$ covariant expressions for the transformation laws (2.5) and (2.7):

$$\delta e_\mu{}^\alpha = -\tfrac{1}{2} i (\bar{\varepsilon}^A \gamma^\alpha \Psi_{\mu A} + \bar{\varepsilon}^{\bar{A}} \gamma^\alpha \Psi_{\mu \bar{A}}) \tag{4.6}$$

$$\delta B_\mu{}^m = -\tfrac{1}{2} e^m{}_A{}^{\bar{B}} [\bar{\varepsilon}^A \Gamma^* \Psi_{\mu \bar{B}} + \tfrac{1}{2} (\bar{\varepsilon} \gamma_\mu \Gamma^b)^A \chi_{b\bar{B}}] + (A \leftrightarrow \bar{A}, a \leftrightarrow \bar{a}). \tag{4.7}$$

Discarding a local $SO(5) \times SO(5)$ rotation we obtain for the supersymmetry transformation of $e^m{}_A{}^{\bar{B}}$ the covariant result

$$\delta e^m{}_A{}^{\bar{B}} = -\tfrac{1}{4} i (\bar{\varepsilon}^C \Gamma_{aC}{}^D \chi_{\bar{b}D} - \bar{\varepsilon}^{\bar{C}} \Gamma_{\bar{b}\bar{C}}{}^{\bar{D}} \chi_{a\bar{D}}) (\Gamma^a e^m \Gamma^{\bar{b}})_A{}^{\bar{B}}. \tag{4.8}$$

One can interpret (4.8) as a (local) $E_{5(+5)} = SO(5, 5)$ rotation in the following way. The spinor representation of $SO(5, 5)$ has 32 components. In the Weyl basis it is sufficient to consider the 16 upper components. We use the explicit form

$$\hat{\Gamma}^{ab}_{A\bar{B},C\bar{D}} = -\Gamma^{ab}_{AC} \delta_{\bar{B}\bar{D}} \qquad \hat{\Gamma}^{\bar{a}\bar{b}}_{A\bar{B},C\bar{D}} = -\Gamma^{\bar{a}\bar{b}}_{\bar{D}\bar{B}} \delta_{AC}$$

$$\hat{\Gamma}^{a\bar{b}}_{A\bar{B},C\bar{D}} = \Gamma^a_{AC} \Gamma^{\bar{b}}_{\bar{D}\bar{B}}. \tag{4.9}$$

Acting on $e^m{}_A{}^{\bar{B}}$ the generators $-\hat{\Gamma}^{ab}_{IJ}$ and $\hat{\Gamma}^{a\bar{b}}_{IJ} (I = A\bar{B}, \ldots)$ give rise to the proper $SO(5) \times SO(5)$ transformation[†]. The introduction of $E_{5(+5)}$ valued fields $(\mathscr{B}, \mathscr{A})$ leads to $SO(5) \times SO(5)$ covariant expressions for the fermionic transformation laws. With the abbreviation

$$F_{\alpha ab} = \varepsilon_{abcde} F_\alpha{}^{cde} \qquad F_a = \varepsilon_{abcde} F^{bcde}$$

one finds

$$\mathscr{B}_{\mu\, ab} = -\tfrac{1}{2} \omega_{\mu\, ab} + (\sqrt{2}/24) F_{\mu\, ab}$$

$$\mathscr{B}_{\mu\, \bar{a}\bar{b}} = -\tfrac{1}{2} \omega_{\mu\, ab} - (\sqrt{2}/24) F_{\mu ab}$$

$$\mathscr{A}_{\mu\, a\bar{b}} = \tfrac{1}{2} \Omega_{\mu(ab)} - \tfrac{1}{4} \eta_{ab} \Omega_{\mu c}{}^c + (\sqrt{2}/24) F_{\mu ab} = -\mathscr{A}_{\mu\, \bar{b}a}$$

$$\mathscr{B}_{m\, ab} = \tfrac{1}{2} e_{[a}{}^n \partial_m e_{b]n} - (\sqrt{2}/192) e_{m[a} F_{b]} \tag{4.10}$$

[†] This representation is in fact sixteen dimensional, because we can impose the $SO(5, 5)$ Majorana condition $e^m_I = \hat{C}_{IJ} (e_J^m)^*$, $\hat{C}_{A\bar{B},C\bar{D}} = C_{5\, AC} C_{5\, \bar{D}\bar{B}}^{-1}$. $e^m{}_{\bar{B}}{}^A \equiv (e^m{}_A{}^{\bar{B}})^*$ transforms in the complex conjugate $SO(5, 5)$ representation. In the gauge (4.5): $e^m{}_{\bar{A}}{}^{\bar{B}} = e^m{}_A{}^{\bar{B}}$.

$$\mathscr{B}_{m\,\bar{a}\bar{b}} = \tfrac{1}{2}e_{[a}{}^n\partial_m e_{b]n} + (\sqrt{2}/192)e_{m[a}\,F_{b]}$$

$$\mathscr{A}_{m\,a\bar{b}} = \tfrac{1}{2}e_{(a}{}^n\partial_m e_{b)n} - \tfrac{1}{4}\eta_{ab}(\Delta^{-1}\partial_m\Delta) - (\sqrt{2}/192)e_{m[a}\,F_{b]}$$

$$\mathscr{B}_{\mu\,IJ} = -\mathscr{B}_{\mu\,ab}\hat{\Gamma}^{ab}_{IJ} + \mathscr{B}_{\mu\,\bar{a}\bar{b}}\hat{\Gamma}^{\bar{a}\bar{b}}_{IJ} \qquad \mathscr{A}_{\mu\,IJ} = \mathscr{A}_{\mu\,a\bar{b}}\hat{\Gamma}^{a\bar{b}}_{IJ} \qquad (\mathscr{B}_m, \mathscr{A}_m \text{ analogous}).$$

The 'generalised vielbein postulate' is

$$\mathscr{D}_\mu e_I^m + \tfrac{1}{4}\partial_n B_\mu{}^n e_I^m + \partial_n B_\mu{}^m e_I^n + (\tfrac{1}{2}\mathscr{B}_{\mu\,IJ} + \mathscr{A}_{\mu\,IJ})e_J^m = 0 \tag{4.11}$$

$$\partial_m e_I^n + (\tfrac{1}{2}\mathscr{B}_{m\,IJ} + \mathscr{A}_{m\,IJ})e_J^n = 0. \tag{4.12}$$

Furthermore, we obtain

$$\mathscr{G}_{\alpha\beta A}{}^{\bar{B}} = -\tfrac{1}{4}\Delta^{1/4}\Omega_{\alpha\beta a}\Gamma^a\Gamma^* + (\sqrt{2}/8)i\Delta^{-1/4}F_{\alpha\beta ab}\Gamma^{ab}$$
$$+ (\sqrt{2}/96)i\Delta^{-1/4}\varepsilon_{\alpha\beta\gamma\delta\varepsilon\zeta}F^{\gamma\delta\varepsilon\zeta}\Gamma^*. \tag{4.13}$$

In addition, there appears a three-index field strength

$$F^+_{\alpha\beta\gamma a} = \tfrac{1}{2}(F_{\alpha\beta\gamma a} + \tfrac{1}{6}\varepsilon_{\alpha\beta\gamma\delta\varepsilon\zeta}F^{\delta\varepsilon\zeta}{}_a)$$
$$F^-_{\alpha\beta\gamma\bar{a}} = \tfrac{1}{2}(F_{\alpha\beta\gamma a} - \tfrac{1}{6}\varepsilon_{\alpha\beta\gamma\delta\varepsilon\zeta}F^{\delta\varepsilon\zeta}{}_a). \tag{4.14}$$

Using $SO(1, 5) \times [SO(5) \times SO(5)]$ covariant derivatives

$$D_\mu\varepsilon_A = [\mathscr{D}_\mu - \tfrac{1}{4}\hat{\omega}_{\mu\,\alpha\beta}\gamma^{\alpha\beta}]\varepsilon_A + \tfrac{1}{2}\mathscr{B}_{\mu\,ab}(\Gamma^{ab}\varepsilon)_A$$
$$D_m\varepsilon_A = [\partial_m + \tfrac{1}{4}(e_\alpha{}^\mu\partial_m e_{\mu\beta})\gamma^{\alpha\beta}]\varepsilon_A + \tfrac{1}{2}\mathscr{B}_{m\,ab}(\Gamma^{ab}\varepsilon)_A \tag{4.15}$$

(and analogous ones for $A \leftrightarrow \bar{A}$, $a \leftrightarrow \bar{a}$) one arrives at manifestly covariant transformation rules for the fermionic fields:

$$\delta\Psi_{\mu A} = [D_\mu(\hat{\omega}, \mathscr{B}_\mu) - \tfrac{1}{8}(\partial_m B_\mu{}^m)]\varepsilon_A + (\tfrac{1}{8}\gamma_\mu\gamma^{\alpha\beta} - e_\mu{}^\alpha\gamma^\beta)\mathscr{G}_{\alpha\beta A}{}^{\bar{B}}\varepsilon_{\bar{B}}$$
$$- \tfrac{1}{4}e^m{}_A{}^{\bar{B}}\Gamma^*[D_m(\gamma_\mu\varepsilon_{\bar{B}}) + (\partial_m g_{\mu\nu})\gamma^\nu\varepsilon_{\bar{B}}]$$
$$+ \tfrac{1}{8}\gamma_\mu A_{ma\bar{b}}(\Gamma^a e^m\Gamma^{\bar{b}})_A{}^{\bar{B}}\varepsilon_{\bar{B}} + (\sqrt{2}/48)i\gamma^{\alpha\beta}\gamma_\mu F^+_{\alpha\beta\gamma d}(\Gamma^d\Gamma^*\varepsilon)_A \tag{4.16}$$

$$\delta\chi_{\bar{a}A} = \tfrac{1}{2}e^m{}_A{}^{\bar{B}}(\Gamma_{\bar{a}}D_m\varepsilon)_{\bar{B}} + \tfrac{1}{4}\gamma^{\alpha\beta}\mathscr{G}_{\alpha\beta A}{}^{\bar{B}}(\Gamma_{\bar{a}}\Gamma^*\varepsilon)_{\bar{B}} - A_{mb\bar{a}}(\Gamma^b e^m)_A{}^{\bar{B}}\Gamma^*\varepsilon_{\bar{B}}$$
$$+ \tfrac{1}{4}A_{mb\bar{c}}(\Gamma^b e^m\Gamma^{\bar{c}})_A{}^{\bar{B}}(\Gamma_{\bar{a}}\Gamma^*\varepsilon)_{\bar{B}} - \gamma^\mu A_{\mu b\bar{a}}(\Gamma^b\varepsilon)_A$$
$$+ (\sqrt{2}/24)i\gamma^{\alpha\beta\gamma}F^-_{\alpha\beta\gamma\bar{a}}\varepsilon_A \tag{4.17}$$

$(A \leftrightarrow \bar{A}, a \leftrightarrow \bar{a}$ for $\delta\Psi_{\mu\bar{A}}$ and $\delta\chi_{a\bar{A}})$.

5. Concluding remarks

The H_{11-D} (and E_{11-D}) structure of $d = 11$ supergravity has now been established for $D = 3, 4, 5, 6$. After a coordinate split the various components of the $d = 11$ graviton and the three-index 'photon' are combined into irreducible representations of the group H_{11-D}. The key ingredient is the generalised vielbein field [4, 5]

$$e^m \sim \Delta^{-s}e^m{}_a\Gamma^a \tag{5.1}$$

which satisfies a 'generalised vielbein postulate' covariant under global E_{11-D} rotations. The fermionic redefinitions coincide with those needed to diagonalise the Rarita-Schwinger terms after dimensional reduction. One obtains H_{11-D} covariant expressions for the supersymmetry transformation laws; the covariance of the equations of motion follows by the usual arguments [4]. There is little doubt that the procedure can also be carried out for the remaining values of D. For $D = 7$ the necessary redefinitions and field assignments can be read off directly from [7].

The general coordinate transformations take a rather unusual form in the covariantised versions of $d = 11$ supergravity. For the elfbein $E_M{}^A$ compensating $SO(1, 10)$ rotations are needed to preserve the gauge (2.1) and a transformation with parameter $\xi^m = -B_\nu{}^m \xi^\nu$ is included [4]. For (5.1) one finds

$$\delta_{gc} e^m = \xi^\mu (\mathscr{D}_\mu e^m + s\partial_n B_\mu^n e^m + \partial_n B_\mu^m e^n) + \xi^n \partial_n e^m - s\partial_n \xi^n e^m - \partial_n \xi^m e^n. \tag{5.2}$$

Comparing with (3.19) and (4.11) we see that $(\mathscr{B}_\mu, \mathscr{A}_\mu)$ generate a D-dimensional general coordinate transformation of e^m. The $(11 - D)$-dimensional part of (5.2) can also be expressed as a E_{11-D} rotation, with parameter $\xi^m (\frac{1}{2}\mathscr{B}_m + \mathscr{A}_m)$ (cf (3.20), (4.12)) and a corresponding term in which $\xi^m (e_a{}^n \partial_m e_{bn})$ is replaced by $e_a{}^m \partial_m \xi^n e_{bn}$.

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