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YOUNG MEASURE SOLUTIONS FOR A CLASS OF FORWARD-BACKWARD CONVECTION-DIFFUSION EQUATIONS

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Abstract. This paper is devoted to the first initial boundary value problems of a class of forward-backward convection-diffusion equations. The existence theorem and the continuous dependence theorem of Young measure solutions are established.

1. Introduction. In this paper we consider the following first initial boundary value problem

$$\frac{\partial u}{\partial t} = \mathrm{div}\vec{\Phi}(\nabla u) + \mathrm{div}\vec{A}(x,t,u) + B(x,t,u), \qquad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with appropriately smooth boundary $\partial\Omega$, $T > 0$, $\vec{\Phi} = \nabla\Psi$ with some potential $\Psi \in C^1(\mathbb{R}^n)$, and Ψ and $\vec{\Phi}$ satisfy

$$\max\{\lambda|\xi|^2 - 1, 0\} \le \Psi(\xi) \le \Lambda|\xi|^2 + 1, \quad |\vec{\Phi}(\xi)| \le \Lambda|\xi|, \quad \xi \in \mathbb{R}^n \quad (0 < \lambda \le \Lambda). \tag{1.4}$$

It is noted that Ψ is not assumed to be convex, in which case the monotonicity condition

$$(\vec{\Phi}(\xi) - \vec{\Phi}(\zeta)) \cdot (\xi - \zeta) \ge 0$$

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is violated for some $\xi, \zeta \in \mathbb{R}^n$. Therefore, (1.1) is a degenerate, forward-backward convection-diffusion equation and admits no classical solutions in general. The non-convexity of the potential is compatible with the usual requirement

$$\vec{\Phi}(\xi) \cdot \xi \geq 0, \quad \xi \in \mathbb{R}^n,$$

which was imposed on the theory of thermal conductors by the Clausius-Duhem inequality ([4]). Further, (1.1) can be strongly degenerate to a first-order hyperbolic equation since $\vec{\Phi}$ may vanish in a bounded domain. Equations like (1.1) arise in modeling the phenomena in melting or freezing when superheating or supercooling occurs and in modeling phase transitions when unstable and metastable states are allowed ([7, 21]).

In the one-dimensional case, if there are no convection or source terms, then (1.1) is simplified into

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \Phi\Big(\frac{\partial u}{\partial x}\Big), \quad (x,t) \in (0,1) \times (0,T). \tag{1.5}$$

The original motivation for studying (1.5) comes from the Clausius-Duhem inequality. For the second initial boundary value problem of (1.5), in the pioneering works of Höllig and Nohel [11] and Höllig [10], infinitely many weak solutions were constructed under the main constitutive assumption that Φ is piecewise affine, which was relaxed in a recent work of Zhang [24]. However, Lair [15] proved that there exists at most one smooth solution. The asymptotic behavior of measure-valued solutions of the first or second initial boundary value problem of (1.5) was studied in Slemrod [17].

Young measure representation was applied to forward-backward diffusion equations in [14], where Kinderlehrer and Pedregal studied (1.1) with $\vec{A} = \vec{0}$ and $B = 0$, i.e.

$$\frac{\partial u}{\partial t} = \operatorname{div} \vec{\Phi}(\nabla u), \quad (x,t) \in Q_T. \tag{1.6}$$

Useful discussions of Young measures were given by Young [23], Tartar [18, 19], DiPerna [6], Ball [1] and Evans [9]. In [14], a Young measure solution $u \in L^\infty((0,T); H_0^1(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(Q_T)$ to the problem (1.6), (1.2) and (1.3) is defined as follows:

$$\frac{\partial u}{\partial t}(x,t) = \operatorname{div} \int_{\mathbb{R}^n} \vec{\Phi}(\xi) d\nu_{x,t}(\xi), \quad \text{in } H^{-1}(Q_T),$$

$$\nabla u(x,t) = \int_{\mathbb{R}^n} \xi d\nu_{x,t}(\xi), \quad \text{a.e. } (x,t) \in Q_T$$

and (1.3) holds in the sense of trace, where $(\nu_{x,t})_{(x,t)\in Q_T}$ is a $W^{1,2}(Q_T)$–gradient Young measure in $\mathbb{R}^n$. Kinderlehrer and Pedregal [14] proved the existence of Young measure solutions to the problem (1.6), (1.2) and (1.3) using Rothe's method and variational method with the relaxation theorem. However, there is no uniqueness theorem in this paper. Demoulini [5] deeply investigated the properties of Young measure solutions obtained in [14] and found that the uniqueness of the Young measure solution is contingent upon the following independence property:

$$\int_{\mathbb{R}^n} \vec{\Phi}(\xi) \cdot \xi d\nu_{x,t}(\xi) = \int_{\mathbb{R}^n} \vec{\Phi}(\xi) d\nu_{x,t}(\xi) \cdot \int_{\mathbb{R}^n} \xi d\nu_{x,t}(\xi), \quad \text{a.e. } (x,t) \in Q_T,$$

namely the heat flux $\vec{\Phi}$ and the gradient ∇u of the solution being independent with respect to the Young measure ν. The asymptotic behavior of Young measure solutions was also studied in [5]. Subsequently, Yin and Wang [22] considered the problem (1.6), (1.2) and (1.3) when $\vec{\Phi}$ is of sublinear growth and when $\vec{\Phi}$ is bounded. Particularly, in the case when $\vec{\Phi}$ is bounded, the equation (1.6) is singular and Young measure solutions should be considered in $L^\infty((0,T);BV(\Omega))$. Moreover, if $\vec{\Phi}$ is taken as

$$\vec{\Phi}(\xi) = \frac{\xi}{1+|\xi|^2}, \quad \xi \in \mathbb{R}^n,$$

(1.6) is just the nonlinear scale-space model in signal processing (one-dimensional case) or image processing (two-dimensional case) proposed by Perona and Malik [16] (see also [2, 8, 12, 25]).

In this paper, we investigate Young measure solutions of the problem (1.1)–(1.3). A motivation arises from image processing. Just like the Perona-Malik model, forward-backward equations can be regarded as a model with the effect of both edge detection and noise removal. It should be noticed that diffusion, either forward or backward, can always destroy the correct position of the edge of an image. However, it is possible to construct an appropriate convection-diffusion model to avoid the appearance of this unexpected phenomenon ([20]).

The structure condition (1.4) shows that (1.1) not only is of forward-backward convection-diffusion type, but also can be strongly degenerate to a first-order hyperbolic equation. Therefore, the convection term can bring some essential differences and difficulties, and we must seek a way to estimate the convection term. In the present paper, we define Young measure solutions of the problem (1.1)–(1.3) and establish its existence and continuous dependence theorems in a similar way as [5, 14]. The existence theorem is proved by using Rothe's method and variational method with the relaxation theorem. As to the continuous dependence theorem, its basis is on an independence property which is satisfied by the Young measure solution constructed in the existence theorem.

The paper is organized as follows. Some preliminaries on Young measure are recalled in Section 2. Subsequently, in Section 3, we show the well-posedness of the problem (1.1)–(1.3), where the existence and the continuous dependence of the Young measure solution is proved.

2. Some preliminaries on Young measure. In this section, let us recall some definitions and results on Young measure.

We use $C_0(\mathbb{R}^n)$ to denote the closure of continuous functions in $\mathbb{R}^n$ with compact supports. The dual of $C_0(\mathbb{R}^n)$ can be identified with the space $\mathscr{M}(\mathbb{R}^n)$ of signed Radon measures with finite mass via the pairing

$$\langle \mu, f\rangle = \int_{\mathbb{R}^n} f d\mu, \quad f \in C_0(\mathbb{R}^n),\ \mu \in \mathscr{M}(\mathbb{R}^n).$$

Let $D \subset \mathbb{R}^n$ or $D \subset \mathbb{R}^n \times \mathbb{R}$ be a measurable set of finite measure. A map $\nu : D \to \mathscr{M}(\mathbb{R}^n)$ is called weakly $*$ measurable if

$$\langle \nu, f\rangle : D \to \mathbb{R},\ x \longmapsto \int_{\mathbb{R}^n} f d\nu_x \quad (\nu_x = \nu(x))$$

is measurable for each $f \in C_0(\mathbb{R}^n)$.

LEMMA 2.1 (Fundamental Theorem on Young Measure). Let $z_k : D \to \mathbb{R}^n$ $(k = 1, 2, \cdots)$ be a sequence of measurable functions. Then there exists a subsequence $\{z_{k_i}\}_{i=1}^{\infty}$ and a weakly $*$ measurable map $\nu : D \to \mathscr{M}(\mathbb{R}^n)$ such that

(i) $\nu(x) \geq 0$, $\|\nu(x)\|_{\mathscr{M}(\mathbb{R}^n)} = \displaystyle\int_{\mathbb{R}^n} d\nu_x \leq 1$, a.e. $x \in D$;

(ii) For each $f \in C_0(\mathbb{R}^n)$, $f(z_{k_i})$ converges weakly $*$ to $\langle \nu, f\rangle$ in $L^\infty(D)$.

Furthermore, one has $\|\nu(x)\|_{\mathscr{M}(\mathbb{R}^n)} = 1$ a.e. $x \in D$ if and only if the subsequence does not escape to infinity.

DEFINITION 2.1. The map $\nu : D \to \mathscr{M}(\mathbb{R}^n)$ in Lemma 2.1 is called the Young measure in $\mathbb{R}^n$ generated by the sequence $\{z_{k_i}\}_{i=1}^{\infty}$.

For $p \geq 1$, define

$$\mathscr{E}_0^p(\mathbb{R}^n) = \left\{\phi \in C(\mathbb{R}^n) : \lim_{|\xi|\to+\infty} \frac{|\phi(\xi)|}{1+|\xi|^p} \text{ exists}\right\},$$

$$\mathscr{E}^p(\mathbb{R}^n) = \left\{\phi \in C(\mathbb{R}^n) : \sup_{\xi\in\mathbb{R}^n} \frac{|\phi(\xi)|}{1+|\xi|^p} < +\infty\right\}.$$

Under the norm

$$\|\phi\|_{\mathscr{E}^p(\mathbb{R}^n)} = \sup_{\xi\in\mathbb{R}^n} \frac{|\phi(\xi)|}{1+|\xi|^p}, \quad \phi \in \mathscr{E}^p(\mathbb{R}^n),$$

$\mathscr{E}_0^p(\mathbb{R}^n)$ is a separable Banach space while $\mathscr{E}^p(\mathbb{R}^n)$ is an inseparable space ([13]).

DEFINITION 2.2. Let $p \geq 1$. A Young measure $\nu = (\nu_x)_{x\in D}$ in $\mathbb{R}^n$ is called a $W^{1,p}(D)$–gradient Young measure if

(i) For each bounded continuous function f in $\mathbb{R}^n$, $\langle \nu, f\rangle$ is measurable in D;

(ii) There is a sequence $\{u_k\}_{k=1}^{\infty} \subset W^{1,p}(D)$ for which the representation formula

$$\lim_{k\to\infty} \int_E \phi(\nabla u_k(x))dx = \int_E \langle \nu_x, \phi\rangle dx$$

holds for any measurable set $E \subset D$ and any function $\phi \in \mathscr{E}_0^p(\mathbb{R}^n)$.

We also say that ν is the $W^{1,p}(D)$–gradient Young measure generated by $\{\nabla u_k\}_{k=1}^{\infty}$ and $\{\nabla u_k\}_{k=1}^{\infty}$ the $W^{1,p}(D)$–gradient generating sequence of ν. In addition, the representation formula also holds for $\phi \in \mathscr{E}^p(\mathbb{R}^n)$, and it follows from Lemma 2.1 that

$$\|\nu(x)\|_{\mathscr{M}(\mathbb{R}^n)} = 1, \quad \text{a.e. } x \in D.$$

We state the following three lemmas, whose proofs can be found in [5, 13, 14, 22].

LEMMA 2.2. Let $\nu = (\nu_x)_{x\in D}$ be a Young measure in $\mathbb{R}^n$. Then $\nu = (\nu_x)_{x\in D}$ is a $W^{1,p}(D)$–gradient Young measure if and only if

(i) There exists $u \in W^{1,p}(D)$ such that

$$\nabla u(x) = \langle \nu_x, \mathrm{id}\rangle, \quad \text{a.e. } x \in D,$$

where id is the unit mapping in $\mathbb{R}^n$;

(ii) For each continuous, quasiconvex and bounded below function $\phi \in \mathscr{E}^p(\mathbb{R}^n)$, the Jensen inequality

$$\phi(\nabla u(x)) \leq \langle \nu_x, \phi\rangle, \quad \text{a.e. } x \in D$$

holds;

(iii) $\langle \nu_x, \phi_p \rangle \in L^1(D)$, where

$$\phi_p(\xi) = |\xi|^p, \quad \xi \in \mathbb{R}^n.$$

LEMMA 2.3. Suppose that $f \in \mathscr{E}^p(\mathbb{R}^n)$ is continuous, quasiconvex and bounded below and that $\{u_k\}_{k=1}^\infty$ converges weakly to u in $W^{1,p}(D)$. Then for any measurable $E \subset D$,

$$\int_E f(\nabla u)dx \le \underline{\lim}_{k\to\infty} \int_E f(\nabla u_k)dx.$$

If, in addition,

$$\lim_{k\to\infty} \int_D f(\nabla u_k)dx = \int_D f(\nabla u)dx,$$

then there exists a subsequence $\{u_{k_i}\}_{i=1}^\infty$ such that $\{f(\nabla u_{k_i})\}_{i=1}^\infty$ converges weakly to $f(\nabla u)$ in $L^1(D)$; furthermore, if

$$\max\{c|\xi|^p - 1, 0\} \le f(\xi) \le C|\xi|^p + 1 \quad (0 < c \le C)$$

then the Young measure ν generated by $\{\nabla u_{k_i}\}_{i=1}^\infty$ is a $W^{1,p}(D)$–gradient Young measure.

LEMMA 2.4. Suppose that $1 \le q < p$ and $\nu^m = (\nu_x^m)_{x\in D}$ is a $W^{1,p}(D)$–gradient Young measure generated by $\{\nabla u^{m,k}\}_{k=1}^\infty$ for $m = 1, 2, \cdots$, where $u^{m,k}$ is uniformly bounded in $W^{1,p}(D)$ with respect to m and k. Then there exist a subsequence of $\{\nu^m\}_{m=1}^\infty$, denoted by $\{\nu^{m_i}\}_{i=1}^\infty$, and a $W^{1,p}(D)$-gradient Young measure ν such that

(i) $\{\nu^{m_i}\}_{i=1}^\infty$ converges to ν weakly $*$ in $L^\infty(D; \mathscr{M}(\mathbb{R}^n))$ and weakly in $L^1(D; (\mathscr{E}_0^q(\mathbb{R}^n))')$;

(ii) $\{\nu^{m_i}\}_{i=1}^\infty$ converges weakly to ν in the biting sense in $L^1(D; (\mathscr{E}_0^p(\mathbb{R}^n))')$. Here, we say that $\{z_i\}_{i=1}^\infty \subset L^1(D)$ converges weakly to $z \in L^1(D)$ in the biting sense in $L^1(D)$, if there is a decreasing sequence of subsets $\{E_j\}_{j=1}^\infty$ of D with $\lim\limits_{j\to\infty} \text{meas} E_j = 0$ such that $\{z_i\}_{i=1}^\infty$ converges weakly to z in $L^1(D\backslash E_j)$ for each $j = 1, 2, \cdots$.

3. Existence and continuous dependence of Young measure solutions. In this section, we investigate the Young measure solution to the problem (1.1)–(1.3) and show the problem is well posed.

3.1. *Definition of Young measure solutions.* Denote Ψ^* the convexification of Ψ. Since $\Psi \in C^1(\mathbb{R}^n)$, $\Psi^* \in C^1(\mathbb{R}^n)$ is convex. Set

$$\vec{\Phi}^* = \nabla \Psi^*.$$

Note that $\vec{\Phi}^* = \vec{\Phi}$ on the set $\{\xi \in \mathbb{R}^n : \Psi(\xi) = \Psi^*(\xi)\}$. Furthermore, Ψ^* and $\vec{\Phi}^*$ satisfy the same structure condition (1.4) as Ψ and $\vec{\Phi}$, i.e.

$$\max\{\lambda|\xi|^2 - 1, 0\} \le \Psi^*(\xi) \le \Lambda|\xi|^2 + 1, \quad |\vec{\Phi}^*(\xi)| \le \Lambda|\xi|, \quad \xi \in \mathbb{R}^n. \tag{3.1}$$

DEFINITION 3.1. A function $u \in L^\infty((0,T); H_0^1(\Omega))$ with $\dfrac{\partial u}{\partial t} \in L^2(Q_T)$ is said to be a Young measure solution to the problem (1.1)–(1.3), if there exists a $W^{1,2}(Q_T)$–gradient

Young measure $\nu = (\nu_{x,t})_{(x,t)\in Q_T}$ in $\mathbb{R}^n$ such that

$$\iint_{Q_T} \Big(\frac{\partial u}{\partial t}\varphi + \langle \nu, \vec{\Phi}\rangle \cdot \nabla\varphi + \vec{A}(x,t,u)\cdot\nabla\varphi - B(x,t,u)\varphi\Big)dxdt = 0 \tag{3.2}$$

for any $\varphi \in L^\infty((0,T); H_0^1(\Omega))$, and

$$\nabla u(x,t) = \langle \nu_{x,t}, \mathrm{id}\rangle, \qquad \text{a.e. } (x,t)\in Q_T, \tag{3.3}$$

$$\int_\Omega \langle \nu_{x,t}, \vec{\Phi}\cdot \mathrm{id}\rangle dx = \int_\Omega \langle \nu_{x,t}, \vec{\Phi}\rangle \cdot \langle \nu_{x,t}, \mathrm{id}\rangle dx, \qquad \text{a.e. } t\in(0,T), \tag{3.4}$$

$$\mathrm{supp}\nu_{x,t} \subset \{\xi\in\mathbb{R}^n : \Psi(\xi) = \Psi^*(\xi)\}, \qquad \text{a.e. } (x,t)\in Q_T, \tag{3.5}$$

and (1.3) holds in the sense of trace.

3.2. *Existence of Young measure solutions.* We first establish the existence theorem. Here, some structure conditions on $\vec{A}$ and B are needed. For convenience, it is assumed that

$$|\vec{A}(x,t,z)| + |\mathrm{div}\vec{A}(x,t,z)| + |B(x,t,z)| \le M(1+|z|), \quad \Big|\frac{\partial \vec{A}}{\partial z}(x,t,z)\Big| \le M \tag{3.6}$$

for each $(x,t,z)\in Q_T\times\mathbb{R}$, where $M>0$.

THEOREM 3.1. Assume that $\vec{A}\in C^{(1,0,1)}(\overline{Q}_T\times\mathbb{R})$ and $B\in C(\overline{Q}_T\times\mathbb{R})$ satisfy (3.6). For any $u_0\in H_0^1(\Omega)$, the problem (1.1)–(1.3) admits at least one Young measure solution.

Proof. We prove the existence theorem by using Rothe's method and variational method with the relaxation theorem.

STEP 1. Solve the difference equations.

Let m be a positive integer. Define

$$\begin{aligned} F_m(v; u_m^{j-1}) = \int_\Omega \Big(\frac{m}{2T}(v-u_m^{j-1})^2 + \Psi(\nabla v) - \Big(&\mathrm{div}\vec{A}\big(x,(j-1)T/m, u_m^{j-1}\big) \\ &+ B\big(x,(j-1)T/m, u_m^{j-1}\big)\Big)v\Big)dx, \quad v\in H_0^1(\Omega), \\ F_m^*(v; u_m^{j-1}) = \int_\Omega \Big(\frac{m}{2T}(v-u_m^{j-1})^2 + \Psi^*(\nabla v) - \Big(&\mathrm{div}\vec{A}\big(x,(j-1)T/m, u_m^{j-1}\big) \\ &+ B\big(x,(j-1)T/m, u_m^{j-1}\big)\Big)v\Big)dx, \quad v\in H_0^1(\Omega), \end{aligned}$$

where $u_m^{j-1}\in H_0^1(\Omega)$, $j=1,2,\cdots,m$. The two functionals are both lower bounded owing to (1.4) and (3.1). Furthermore, it follows from the relaxation theorem ([3]) that

$$\inf\{F_m(v;u_m^{j-1}) : v\in H_0^1(\Omega)\} = \inf\{F_m^*(v;u_m^{j-1}) : v\in H_0^1(\Omega)\}. \tag{3.7}$$

Let $\{u_m^{j,k}\}_{k=1}^\infty \subset H_0^1(\Omega)$ be a minimizing sequence of $F_m(\cdot; u_m^{j-1})$ satisfying

$$F_m(u_m^{j,k}; u_m^{j-1}) = \min_{\theta\in\mathbb{R}} F_m(\theta u_m^{j,k}; u_m^{j-1}), \quad k=1,2,\cdots, \tag{3.8}$$

$$F_m(u_m^{j,k}; u_m^{j-1}) < \inf\{F_m(v;u_m^{j-1}) : v\in H_0^1(\Omega)\} + \frac{1}{mk}, \quad k=1,2,\cdots. \tag{3.9}$$

Since $\Psi^*(\cdot)\le\Psi(\cdot)$, (3.7) and (3.9) yield

$$F_m^*(u_m^{j,k}; u_m^{j-1}) < \inf\{F_m^*(v;u_m^{j-1}) : v\in H_0^1(\Omega)\} + \frac{1}{mk}, \quad k=1,2,\cdots, \tag{3.10}$$

which, together with the Hölder inequality, implies that

$$\begin{aligned}
&\int_\Omega \Big(\frac{m}{2T}(u_m^{j,k}-u_m^{j-1})^2+\Psi^*(\nabla u_m^{j,k})\Big)dx-\frac{1}{mk}\\
\le&\int_\Omega \Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)+B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)u_m^{j,k}dx+F_m^*(u_m^{j-1};u_m^{j-1})\\
\le&\int_\Omega \Big(\Psi^*(\nabla u_m^{j-1})+\Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)+B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)\\
&\cdot(u_m^{j,k}-u_m^{j-1})\Big)dx\\
\le&\int_\Omega \Psi^*(\nabla u_m^{j-1})dx+\frac{T}{m}\int_\Omega\Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)+B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)^2dx\\
&\qquad+\frac{m}{4T}\int_\Omega (u_m^{j,k}-u_m^{j-1})^2dx,\quad k=1,2,\cdots.
\end{aligned}$$

Using this formula, (3.6) and the Poincaré inequality, one gets that

$$\begin{aligned}
&\int_\Omega \Big(\frac{m}{4T}(u_m^{j,k}-u_m^{j-1})^2+\Psi^*(\nabla u_m^{j,k})\Big)dx\\
\le&\int_\Omega \Psi^*(\nabla u_m^{j-1})dx+\frac{T}{m}\int_\Omega\Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)+B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)^2dx\\
&+\frac{1}{mk}\\
\le&\int_\Omega \Psi^*(\nabla u_m^{j-1})dx+\frac{C_1}{m}\int_\Omega(1+|u_m^{j-1}|^2+|\nabla u_m^{j-1}|^2)dx+\frac{1}{mk}\\
\le&\int_\Omega \Psi^*(\nabla u_m^{j-1})dx+\frac{C_2}{m}\int_\Omega|\nabla u_m^{j-1}|^2dx+\frac{C_2}{m}\\
\le&\Big(1+\frac{C_0}{m}\Big)\int_\Omega \Psi^*(\nabla u_m^{j-1})dx+\frac{C_0}{m},\quad j=1,2,\cdots,m,\ k=1,2,\cdots,
\end{aligned}\tag{3.11}$$

where C_1, C_2, and C_0 depend only on Ω, T, λ, Λ and M.

Take $u_m^0=u_0$. Then (3.11) with $j=1$ shows that $\{u_m^{1,k}\}_{k=1}^\infty$ is uniformly bounded in $H_0^1(\Omega)$. Hence there exist a convergent subsequence of $\{u_m^{1,k}\}_{k=1}^\infty$ and a function $u_m^1\in H_0^1(\Omega)$, such that

$$\nabla u_m^{1,k}\rightharpoonup\nabla u_m^1 \text{ weakly in } L^2(\Omega) \text{ and } u_m^{1,k}\to u_m^1 \text{ strongly in } L^2(\Omega) \text{ as } k\to\infty.\tag{3.12}$$

Here and in the proof of this theorem, we always denote a convergent subsequence of a sequence by itself for convenience. It follows from (3.10) and (3.12) that

$$\int_\Omega \Psi^*(\nabla u_m^1)dx=\lim_{k\to\infty}\int_\Omega \Psi^*(\nabla u_m^{1,k})dx,$$
$$\int_\Omega\Big(\frac{m}{4T}(u_m^1-u_m^0)^2+\Psi^*(\nabla u_m^1)\Big)dx\le\Big(1+\frac{C_0}{m}\Big)\int_\Omega\Psi^*(\nabla u_m^0)dx+\frac{C_0}{m}.$$

Repeating the above process in turn, we get $u_m^j\in H_0^1(\Omega)$ for $j=1,2,\cdots,m$, which satisfies

$$\nabla u_m^{j,k}\rightharpoonup\nabla u_m^j \text{ weakly in } L^2(\Omega) \text{ and } u_m^{j,k}\to u_m^j \text{ strongly in } L^2(\Omega) \text{ as } k\to\infty,\tag{3.13}$$

$$\int_\Omega \Psi^*(\nabla u_m^j)dx = \lim_{k\to\infty}\int_\Omega \Psi^*(\nabla u_m^{j,k})dx, \tag{3.14}$$

$$\int_\Omega \Big(\frac{m}{4T}(u_m^j - u_m^{j-1})^2 + \Psi^*(\nabla u_m^j)\Big)dx \le \Big(1+\frac{C_0}{m}\Big)\int_\Omega \Psi^*(\nabla u_m^{j-1})dx + \frac{C_0}{m}. \tag{3.15}$$

Direct calculation shows that

$$\Big(1+\frac{C_0}{m}\Big)^m \le \mathrm{e}^{C_0}, \quad \sum_{i=0}^{m-1}\Big(1+\frac{C_0}{m}\Big)^i \le \frac{m}{C_0}\mathrm{e}^{C_0}.$$

Then, one gets from (3.11) and (3.15) that

$$\int_\Omega \Psi^*(\nabla u_m^{j,k})dx \le \mathrm{e}^{C_0}\int_\Omega \Psi^*(\nabla u_0)dx + \mathrm{e}^{C_0}, \quad j=1,2,\cdots,m,\ k=1,2,\cdots, \tag{3.16}$$

$$\int_\Omega \Psi^*(\nabla u_m^j)dx \le \mathrm{e}^{C_0}\int_\Omega \Psi^*(\nabla u_0)dx + \mathrm{e}^{C_0}, \quad j=1,2,\cdots,m. \tag{3.17}$$

Summing up (3.15) from 1 to m leads to

$$\frac{m}{4T}\sum_{j=1}^m\int_\Omega (u_m^j-u_m^{j-1})^2dx + \sum_{j=1}^m\int_\Omega \Psi^*(\nabla u_m^j)dx \le \Big(1+\frac{C_0}{m}\Big)\sum_{j=1}^m\int_\Omega \Psi^*(\nabla u_m^{j-1})dx + C_0,$$

which, together with (3.17), implies that

$$\begin{aligned}\frac{m}{4T}\sum_{j=1}^m\int_\Omega (u_m^j-u_m^{j-1})^2dx \le& \frac{C_0}{m}\sum_{j=1}^m\int_\Omega \Psi^*(\nabla u_m^{j-1})dx + \int_\Omega \Psi^*(\nabla u_m^0)dx + C_0\\ \le& \Big(C_0\mathrm{e}^{C_0}+1\Big)\int_\Omega \Psi^*(\nabla u_0)dx + C_0\mathrm{e}^{C_0} + C_0.\end{aligned} \tag{3.18}$$

For $j=1,2,\cdots,m$, denote $\nu^{m,j} = (\nu_x^{m,j})_{x\in\Omega}$ the Young measure generated by $\{\nabla u_m^{j,k}\}_{k=1}^\infty$. By Lemma 2.3 with (3.13) and (3.14), $\nu^{m,j}$ is a $W^{1,2}(\Omega)$–gradient Young measure and

$$\nabla u_m^j(x) = \lim_{k\to\infty}\nabla u_m^{j,k}(x) = \langle \nu_x^{m,j}, \mathrm{id}\rangle, \quad \text{a.e. } x\in\Omega. \tag{3.19}$$

Since $\Psi^*(\cdot)\le\Psi(\cdot)$, one gets from (3.7), (3.9), (3.10) and (3.13) that

$$\lim_{k\to\infty}\int_\Omega \Psi^*(\nabla u_m^{j,k})dx = \lim_{k\to\infty}\int_\Omega \Psi(\nabla u_m^{j,k})dx, \quad j=1,2,\cdots,m.$$

Therefore,

$$\int_\Omega \langle \nu^{m,j}, \Psi^*\rangle dx = \int_\Omega \langle \nu^{m,j}, \Psi\rangle dx, \quad j=1,2,\cdots,m,$$

which implies

$$\mathrm{supp}\nu_x^{m,j} \subset \{\xi\in\mathbb{R}^n : \Psi(\xi)=\Psi^*(\xi)\}, \quad \text{a.e. } x\in\Omega, \quad j=1,2,\cdots,m. \tag{3.20}$$

Now we deduce the equilibrium equation. Fix $j=1,2,\cdots,m$. It follows from (3.10), (3.13) and (3.14) that u_m^j is just a minimum of $F_m^*(\cdot;u_m^{j-1})$. Take the Gâteaux derivative to yield

$$\int_\Omega \Big(\frac{m}{T}(u_m^j-u_m^{j-1})\eta + \vec{\Phi}^*(\nabla u_m^j)\cdot\nabla\eta - \Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)$$

$$+ B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)\eta\Big)dx = 0, \quad \eta \in H_0^1(\Omega). \tag{3.21}$$

Since

$$F_m^*(u_m^j;u_m^{j-1}) \le F_m^*(u_m^j+\varepsilon\eta;u_m^{j-1}), \quad \eta \in H_0^1(\Omega),\ \varepsilon\in\mathbb{R},$$

one can derive that

$$\int_\Omega \Big(\frac{m}{T}(u_m^j-u_m^{j-1})\eta + \langle \nu^{m,j},\vec{\Phi}^*\rangle\cdot\nabla\eta - \Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)$$

$$+ B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)\eta\Big)dx = 0, \quad \eta \in H_0^1(\Omega). \tag{3.22}$$

Then, (3.20)–(3.22) show that

$$\langle \nu_x^{m,j},\vec{\Phi}\rangle = \langle \nu_x^{m,j},\vec{\Phi}^*\rangle = \vec{\Phi}^*(\nabla u_m^j(x)), \quad \text{a.e. } x\in\Omega, \quad j=1,2,\cdots,m \tag{3.23}$$

and

$$\int_\Omega \Big(\frac{m}{T}(u_m^j-u_m^{j-1})\eta + \langle \nu^{m,j},\vec{\Phi}\rangle\cdot\nabla\eta + \vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)\cdot\nabla\eta$$

$$- B\big(x,(j-1)T/m,u_m^{j-1}\big)\eta\Big)dx = 0, \quad \eta\in H_0^1(\Omega), \quad j=1,2,\cdots,m. \tag{3.24}$$

Below we derive the following independence property

$$\int_\Omega \langle \nu_x^{m,j},\vec{\Phi}\cdot\mathrm{id}\rangle dx = \int_\Omega \langle \nu_x^{m,j},\vec{\Phi}\rangle\cdot\langle \nu_x^{m,j},\mathrm{id}\rangle dx, \quad j=1,2,\cdots,m. \tag{3.25}$$

For $j=1,2,\cdots,m$, (3.8) implies

$$\int_\Omega \vec{\Phi}(\nabla u_m^{j,k})\cdot\nabla u_m^{j,k}dx = \int_\Omega \Big(-\frac{m}{T}(u_m^{j,k}-u_m^{j-1})u_m^{j,k} + \Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)$$

$$+ B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)u_m^{j,k}\Big)dx, \quad k=1,2,\cdots. \tag{3.26}$$

Rewrite (3.24) with $\eta = u_m^j$ into

$$\int_\Omega \langle \nu^{m,j},\vec{\Phi}\rangle\cdot\nabla u_m^j dx = \int_\Omega \Big(-\frac{m}{T}(u_m^j-u_m^{j-1})u_m^j + \Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big)$$

$$+ B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)u_m^j\Big)dx. \tag{3.27}$$

Then, it follows from (3.26), (3.23), (3.19) and (3.21) with $\eta = u_m^j$ that

$$\int_\Omega \langle \nu^{m,j},\vec{\Phi}\cdot\mathrm{id}\rangle dx - \int_\Omega \langle \nu^{m,j},\vec{\Phi}\rangle\cdot\langle \nu^{m,j},\mathrm{id}\rangle dx$$

$$= \lim_{k\to\infty}\int_\Omega \vec{\Phi}(\nabla u_m^{j,k})\cdot\nabla u_m^{j,k}dx - \int_\Omega \langle \nu^{m,j},\vec{\Phi}\rangle\cdot\nabla u_m^j dx$$

$$= -\lim_{k\to\infty}\int_\Omega \Big(\frac{m}{T}(u_m^{j,k}-u_m^{j-1})u_m^{j,k} - \frac{m}{T}(u_m^j-u_m^{j-1})u_m^j\Big)dx$$

$$+ \lim_{k\to\infty}\int_\Omega \Big(\mathrm{div}\vec{A}\big(x,(j-1)T/m,u_m^{j-1}\big) + B\big(x,(j-1)T/m,u_m^{j-1}\big)\Big)\big(u_m^{j,k}-u_m^j\big)dx$$

$$=0,$$

which, together with (3.13), leads to (3.25).

Step 2. Construct the approximate solutions.

Let m be the positive integer given in Step 1. For $j=1,2,\cdots,m$, denote χ_m^j the characteristic function of $\big[(j-1)T/m, jT/m\big)$ and denote

$$\gamma_m^j(t)=\Big(\frac{mt}{T}-(j-1)\Big)\chi_m^j(t),\quad 0\le t\le T.$$

Define

$$u_m(x,t)=\sum_{j=1}^m \chi_m^j(t)u_m^{j-1}(x)+\gamma_m^j(t)(u_m^j(x)-u_m^{j-1}(x)),\qquad (x,t)\in Q_T,$$

$$w_m(x,t)=\sum_{j=1}^m \chi_m^j(t)u_m^j(x),\qquad (x,t)\in Q_T,$$

$$w_m^k(x,t)=\sum_{j=1}^m \chi_m^j(t)u_m^{j,k}(x),\qquad (x,t)\in Q_T,\ k=1,2,\cdots,$$

$$\vec{A}_m(x,t,z)=\sum_{j=1}^m \chi_m^j(t)\vec{A}\big(x,(j-1)T/m,z\big),\qquad (x,t,z)\in Q_T\times\mathbb{R},$$

$$B_m(x,t,z)=\sum_{j=1}^m \chi_m^j(t)B\big(x,(j-1)T/m,z\big),\qquad (x,t,z)\in Q_T\times\mathbb{R}$$

and

$$\nu^m=(\nu_{x,t}^m)_{(x,t)\in Q_T},\quad \nu_{x,t}^m=\sum_{j=1}^m \chi_m^j(t)\nu_x^{m,j},\quad (x,t)\in Q_T.$$

From the definitions of u_m^j, $u_m^{j,k}$ and $\nu^{m,j}$, we get that $u_m, w_m, w_m^k\in L^\infty((0,T);H_0^1(\Omega))$ with $\dfrac{\partial u_m}{\partial t}\in L^2(Q_T)$ and $\nu^m\in L^\infty((0,T);(\mathscr{E}^2(\mathbb{R}^n))')$ is the $W^{1,2}(Q_T)$–gradient Young measure generated by $\{w_m^k\}_{k=1}^\infty$. It follows from (3.1), (3.16)–(3.18) that

$$\|u_m\|_{L^\infty((0,T);H_0^1(\Omega))}\le C,\quad \|w_m\|_{L^\infty((0,T);H_0^1(\Omega))}\le C,\quad \|w_m^k\|_{L^\infty((0,T);H_0^1(\Omega))}\le C,\tag{3.28}$$

$$\Big\|\frac{\partial u_m}{\partial t}\Big\|_{L^2(Q_T)}=\Big(\frac{m}{T}\sum_{j=1}^m\int_\Omega (u_m^j(x)-u_m^{j-1}(x))^2dx\Big)^{1/2}\le C,\tag{3.29}$$

$$\|u_m-w_m\|_{L^2(Q_T)}\le\Big(\frac{T}{m}\sum_{j=1}^m\int_\Omega (u_m^j(x)-u_m^{j-1}(x))^2dx\Big)^{1/2}\le\frac{C}{m},\tag{3.30}$$

where $C>0$ depends only on Ω, T, λ, Λ and M. In addition, (3.19), (3.20) and (3.25) yield

$$\nabla w_m(x,t)=\langle\nu_{x,t}^m,\mathrm{id}\rangle,\quad \text{a.e. } (x,t)\in Q_T,\tag{3.31}$$

$$\mathrm{supp}\nu_{x,t}^m\subset\{\xi\in\mathbb{R}^n:\Psi(\xi)=\Psi^*(\xi)\},\quad \text{a.e. } (x,t)\in Q_T,\tag{3.32}$$

$$\int_\Omega\langle\nu_{x,t}^m,\vec{\Phi}\cdot\mathrm{id}\rangle dx=\int_\Omega\langle\nu_{x,t}^m,\vec{\Phi}\rangle\cdot\langle\nu_{x,t}^m,\mathrm{id}\rangle dx,\quad \text{a.e. } t\in(0,T).\tag{3.33}$$

Furthermore, one can get from (3.24) that

$$\iint_{Q_T}\Big(\frac{\partial u_m}{\partial t}\varphi+\langle\nu^m,\vec{\Phi}\rangle\cdot\nabla\varphi+\vec{A}_m(x,t,w_m)\cdot\nabla\varphi-B_m(x,t,w_m)\varphi\Big)dxdt=0,$$

$$\varphi \in L^\infty((0,T);H_0^1(\Omega)). \quad (3.34)$$

STEP 3. Complete the limiting process.

Owing to (3.28) and (3.29), there exist $u, w \in L^\infty((0,T);H_0^1(\Omega))$ with $\dfrac{\partial u}{\partial t} \in L^2(Q_T)$ such that

$$\nabla u_m \rightharpoonup \nabla u,\ \frac{\partial u_m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t},\ \nabla w_m \rightharpoonup \nabla w,\ w_m \rightharpoonup w \text{ weakly in } L^2(Q_T)$$
$$\text{and } u_m \to u \text{ strongly in } L^2(Q_T) \text{ as } m \to \infty. \quad (3.35)$$

Noting (3.30) implies

$$\lim_{m\to\infty} \|u_m - w_m\|_{L^2(Q_T)} = 0,$$

one can subsequently get that

$$u(x,t) = w(x,t), \quad \text{a.e. } (x,t) \in Q_T. \quad (3.36)$$

Then, (3.35), (3.36) and (3.6) yield that

$$\vec{A}_m(x,t,w_m(x,t)) \to \vec{A}(x,t,u(x,t)),\ B_m(x,t,w_m(x,t)) \to B(x,t,u(x,t))$$
$$\text{strongly in } L^2(Q_T) \text{ as } m \to \infty. \quad (3.37)$$

Due to Lemma 2.4 with (3.28), there exists a $W^{1,2}(Q_T)$–gradient Young measure $\nu \in L^\infty((0,T);(\mathscr{E}^2(\mathbb{R}^n))')$ such that

$$\nu^m \rightharpoonup \nu \quad \text{weakly} * \text{ in } L^\infty(Q_T;\mathscr{M}(\mathbb{R}^n)),\ \text{weakly in } L^1(Q_T;(\mathscr{E}_0^1(\mathbb{R}^n))')$$
$$\text{and weakly in the biting sense in } L^1(Q_T;(\mathscr{E}_0^2(\mathbb{R}^n))') \text{ as } m \to \infty. \quad (3.38)$$

Now we are ready to verify that u with ν is a Young measure solution to the problem (1.1)–(1.3). First, letting $m \to \infty$ in (3.34) with (3.35)–(3.38) leads to (3.2) for any $\varphi \in L^\infty((0,T);H_0^1(\Omega))$. Second, (3.3) follows from (3.31), (3.35), (3.36) and (3.38). Third, (3.34) implies

$$\mathrm{div}\langle \nu^m, \vec{\Phi}\rangle = \frac{\partial u_m}{\partial t} - \mathrm{div}\vec{A}_m(x,t,w_m) - B_m(x,t,w_m)$$

in the sense of distribution in Q_T, which, together with (3.28), (3.29) and (3.6), shows that $\mathrm{div}\langle \nu^m, \vec{\Phi}\rangle \in L^2(Q_T)$ satisfies

$$\|\mathrm{div}\langle \nu^m, \vec{\Phi}\rangle\|_{L^2(Q_T)} \le C, \quad m = 1,2,\cdots, \quad (3.39)$$

where $C > 0$ depends only on Ω, T, λ, Λ and M. Additionally, (3.31) gives

$$\mathrm{curl}\langle \nu^m, \mathrm{id}\rangle = \mathrm{curl}\nabla w_m = 0, \quad \text{a.e. } (x,t) \in Q_T, \quad m = 1,2,\cdots. \quad (3.40)$$

Using the div-curl lemma with (3.38)–(3.40), one gets

$$\langle \nu^m_{x,t}, \vec{\Phi}\rangle \cdot \langle \nu^m_{x,t}, \mathrm{id}\rangle \rightharpoonup \langle \nu_{x,t}, \vec{\Phi}\rangle \cdot \langle \nu_{x,t}, \mathrm{id}\rangle \text{ weakly in } L^1(Q_T) \text{ as } m \to \infty. \quad (3.41)$$

Moreover, (3.38) implies

$$\langle \nu^m_{x,t}, \vec{\Phi}\cdot \mathrm{id}\rangle \rightharpoonup \langle \nu_{x,t}, \vec{\Phi}\cdot \mathrm{id}\rangle \text{ weakly in the biting sense in } L^1(Q_T) \text{ as } m \to \infty. \quad (3.42)$$

Then, (3.4) follows from (3.33), (3.41) and (3.42). Finally, (3.5) can be deduced from (3.32) and (3.38), while (3.29) guarantees that (1.3) holds in the sense of trace. □

3.3. *Continuous dependence of Young measure solutions.* Let us turn to the continuous dependence theorem. As mentioned in the introduction, there is no uniqueness result for the forward-backward equation in general. Here, the independence property (3.4) plays the important and key role in the uniqueness theorem. Furthermore, there also needs to be some structure conditions on $\vec{A}$ and B such that the convection and source terms can be controlled.

THEOREM 3.2. Assume that $\vec{A}$ and B satisfy

$$\vec{A}(x,t,z)=\vec{a}(x,t)z,\quad |\mathrm{div}\vec{a}(x,t)|\le M,\quad \Big|\frac{\partial B(x,t,z)}{\partial z}\Big|\le M,\quad (x,t,z)\in Q_T\times\mathbb{R}$$

with some $M>0$. Let $u,v\in L^\infty((0,T);H_0^1(\Omega))$ with $\dfrac{\partial u}{\partial t},\dfrac{\partial v}{\partial t}\in L^2(Q_T)$ be the Young measure solutions to the problem (1.1)–(1.3) corresponding to the initial data u_0,v_0, respectively. Then there exists $C>0$ depending only on $\|\mathrm{div}\vec{a}\|_{L^\infty(Q_T)}$ and $\Big\|\dfrac{\partial B}{\partial z}\Big\|_{L^\infty(Q_T\times\mathbb{R})}$ such that

$$\int_\Omega (u(x,t)-v(x,t))^2dx\le \mathrm{e}^{Ct}\int_\Omega (u_0(x)-v_0(x))^2dx,\quad 0\le t\le T. \tag{3.43}$$

Proof. Let ν and μ be the $W^{1,2}(Q_T)$–gradient Young measure with respect to u and v, respectively. For any $s\in[0,T]$, choosing

$$\varphi(x,t)=(u(x,t)-v(x,t))\chi_{[0,s]}(t),\quad (x,t)\in Q_T$$

in (3.2), one gets that

$$\iint_{Q_s}\Big(\frac{\partial u}{\partial t}(u-v)+\langle\nu,\vec{\Phi}\rangle\cdot\nabla(u-v)+\vec{a}(x,t)u\cdot\nabla(u-v)-B(x,t,u)(u-v)\Big)dxdt=0$$

and

$$\iint_{Q_s}\Big(\frac{\partial v}{\partial t}(u-v)+\langle\mu,\vec{\Phi}\rangle\cdot\nabla(u-v)+\vec{a}(x,t)v\cdot\nabla(u-v)-B(x,t,v)(u-v)\Big)dxdt=0.$$

Combine these two equalities to get that

$$\begin{aligned}\iint_{Q_s}\Big(&\frac{1}{2}\frac{\partial}{\partial t}(u-v)^2+\big(\langle\nu,\vec{\Phi}\rangle-\langle\mu,\vec{\Phi}\rangle\big)\cdot\nabla(u-v)\\&+\vec{a}(x,t)(u-v)\cdot\nabla(u-v)-\big(B(x,t,u)-B(x,t,v)\big)(u-v)\Big)dxdt=0,\end{aligned}$$

i.e.

$$\begin{aligned}&\int_\Omega (u(x,s)-v(x,s))^2dx-\int_\Omega (u_0(x)-v_0(x))^2dx\\&=-2\iint_{Q_s}\big(\langle\nu,\vec{\Phi}\rangle-\langle\mu,\vec{\Phi}\rangle\big)\cdot\nabla(u-v)dxdt-\iint_{Q_s}\vec{a}(x,t)\cdot\nabla(u-v)^2dxdt\\&\quad+2\iint_{Q_s}\big(B(x,t,u)-B(x,t,v)\big)(u-v)dxdt.\end{aligned} \tag{3.44}$$

We estimate the terms on the right side of (3.44). On the one hand, it follows from (3.3)–(3.5) that

$$\begin{aligned}
&\iint_{Q_s} \big(\langle\nu,\vec{\Phi}\rangle - \langle\mu,\vec{\Phi}\rangle\big)\cdot\nabla(u-v)dxdt\\
=&\iint_{Q_s} \big(\langle\nu,\vec{\Phi}\rangle - \langle\mu,\vec{\Phi}\rangle\big)\cdot\big(\langle\nu,\mathrm{id}\rangle - \langle\mu,\mathrm{id}\rangle\big)dxdt\\
=&\iint_{Q_s} \big(\langle\nu,\vec{\Phi}\rangle\cdot\langle\nu,\mathrm{id}\rangle - \langle\nu,\vec{\Phi}\rangle\cdot\langle\mu,\mathrm{id}\rangle - \langle\mu,\vec{\Phi}\rangle\cdot\langle\nu,\mathrm{id}\rangle + \langle\mu,\vec{\Phi}\rangle\cdot\langle\mu,\mathrm{id}\rangle\big)dxdt\\
=&\iint_{Q_s} \big(\langle\nu,\vec{\Phi}\cdot\mathrm{id}\rangle - \langle\nu,\vec{\Phi}\rangle\cdot\langle\mu,\mathrm{id}\rangle - \langle\mu,\vec{\Phi}\rangle\cdot\langle\nu,\mathrm{id}\rangle + \langle\mu,\vec{\Phi}\cdot\mathrm{id}\rangle\big)dxdt\\
=&\iint_{Q_s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\vec{\Phi}(\xi)-\vec{\Phi}(\zeta)\big)\cdot(\xi-\zeta)d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt\\
=&\iint_{Q_s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\vec{\Phi}^*(\xi)-\vec{\Phi}^*(\zeta)\big)\cdot(\xi-\zeta)d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt\\
=&\iint_{Q_s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\nabla\Psi^*(\xi)-\nabla\Psi^*(\zeta)\big)\cdot(\xi-\zeta)d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt, \qquad (3.45)
\end{aligned}$$

which implies

$$\iint_{Q_s} \big(\langle\nu,\vec{\Phi}\rangle - \langle\mu,\vec{\Phi}\rangle\big)\cdot\nabla(u-v)dxdt \ge 0 \qquad (3.46)$$

since Ψ^* is quasiconvex. On the other hand, integrating the second term on the right side of (3.44), one gets that

$$\begin{aligned}
&-\iint_{Q_s}\vec{a}(x,t)\cdot\nabla(u-v)^2dxdt + 2\iint_{Q_s}\big(B(x,t,u)-B(x,t,v)\big)(u-v)dxdt\\
=&\iint_{Q_s}\mathrm{div}\vec{a}(x,t)(u-v)^2dxdt + 2\iint_{Q_s}\int_0^1\frac{\partial B}{\partial z}(x,t,\sigma u+(1-\sigma)v)d\sigma(u-v)^2dxdt\\
\le&\sup_{Q_T}\Big(\mathrm{div}\vec{a}(x,t)+2\int_0^1\frac{\partial B}{\partial z}(x,t,\sigma u(x,t)+(1-\sigma)v(x,t))d\sigma\Big)\iint_{Q_s}(u-v)^2dxdt\\
\le&\Big(\|\mathrm{div}\vec{a}\|_{L^\infty(Q_T)}+2\Big\|\frac{\partial B}{\partial z}\Big\|_{L^\infty(Q_T\times\mathbb{R})}\Big)\iint_{Q_s}(u-v)^2dxdt. \qquad (3.47)
\end{aligned}$$

Substituting (3.46) and (3.47) into (3.44), one obtains

$$\begin{aligned}
&\int_\Omega(u(x,s)-v(x,s))^2dx - \int_\Omega(u_0(x)-v_0(x))^2dx\\
\le&\Big(\|\mathrm{div}\vec{a}\|_{L^\infty(Q_T)}+2\Big\|\frac{\partial B}{\partial z}\Big\|_{L^\infty(Q_T\times\mathbb{R})}\Big)\iint_{Q_s}(u(x,t)-v(x,t))^2dxdt, \quad s\in[0,T],
\end{aligned}$$

which leads to (3.43) by using the Gronwall inequality. □

In Theorem 3.2, $\vec{A}$ is linear, which can be relaxed if Ψ is strictly convex in the set where $\Psi^* = \Psi$.

THEOREM 3.3. Assume that

$$\big(\nabla\Psi(\xi)-\nabla\Psi(\zeta)\big)\cdot(\xi-\zeta)\ge\delta(\xi-\zeta)^2, \quad \xi,\zeta\in G, \qquad (\delta>0) \qquad (3.48)$$

$$\Big|\frac{\partial \vec{A}(x,t,z)}{\partial z}\Big| \le M, \quad \Big|\frac{\partial B(x,t,z)}{\partial z}\Big| \le M, \quad (x,t,z)\in Q_T\times\mathbb{R}, \qquad (M>0) \tag{3.49}$$

where $G=\{\xi\in\mathbb{R}^n : \Psi(\xi)=\Psi^*(\xi)\}$. Let $u,v\in L^\infty((0,T);H_0^1(\Omega))$ with $\dfrac{\partial u}{\partial t},\dfrac{\partial v}{\partial t}\in L^2(Q_T)$ be the Young measure solutions to the problem (1.1)–(1.3) corresponding to the initial data u_0,v_0, respectively. Then there exists $C>0$ depending only on δ and M such that

$$\int_\Omega (u(x,t)-v(x,t))^2dx \le \mathrm{e}^{Ct}\int_\Omega (u_0(x)-v_0(x))^2dx, \quad 0\le t\le T. \tag{3.50}$$

Proof. Let ν and μ be the $W^{1,2}(Q_T)$–gradient Young measure with respect to u and v, respectively. Similar to the proof of (3.44), one can get that

$$\begin{aligned}
&\int_\Omega (u(x,s)-v(x,s))^2dx-\int_\Omega (u_0(x)-v_0(x))^2dx\\
=&-2\iint_{Q_s}\big(\langle\nu,\vec{\Phi}\rangle-\langle\mu,\vec{\Phi}\rangle\big)\cdot\nabla(u-v)dxdt\\
&\quad-2\iint_{Q_s}(\vec{A}(x,t,u)-\vec{A}(x,t,v))\cdot(\nabla u-\nabla v)dxdt\\
&\quad+2\iint_{Q_s}\big(B(x,t,u)-B(x,t,v)\big)(u-v)dxdt.
\end{aligned}\tag{3.51}$$

Let us estimate each term on the right side of (3.51). First, it follows from (3.45), (3.4) and (3.48) that

$$\begin{aligned}
&\iint_{Q_s}\big(\langle\nu,\vec{\Phi}\rangle-\langle\mu,\vec{\Phi}\rangle\big)\cdot\nabla(u-v)dxdt\\
=&\iint_{Q_s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(\nabla\Psi^*(\xi)-\nabla\Psi^*(\zeta)\big)\cdot(\xi-\zeta)d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt\\
=&\iint_{Q_s}\int_G\int_G\big(\nabla\Psi^*(\xi)-\nabla\Psi^*(\zeta)\big)\cdot(\xi-\zeta)d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt\\
\ge&\delta\iint_{Q_s}\int_G\int_G|\xi-\zeta|^2d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt.
\end{aligned}\tag{3.52}$$

Second, using (3.3), (3.4), (3.49) and the Hölder inequality yields

$$\begin{aligned}
&-\iint_{Q_s}(\vec{A}(x,t,u)-\vec{A}(x,t,v))\cdot(\nabla u-\nabla v)dxdt\\
=&-\iint_{Q_s}(\vec{A}(x,t,u)-\vec{A}(x,t,v))\cdot\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\xi-\zeta)d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt\\
=&-\iint_{Q_s}(\vec{A}(x,t,u)-\vec{A}(x,t,v))\cdot\int_G\int_G(\xi-\zeta)d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)dxdt\\
\le&\frac{1}{4\delta}\iint_{Q_s}|\vec{A}(x,t,u)-\vec{A}(x,t,v))|^2dxdt+\delta\iint_{Q_s}\Big(\int_G\int_G|\xi-\zeta|d\nu_{x,t}(\xi)d\mu_{x,t}(\zeta)\Big)^2dxdt\\
\le&\frac{1}{4\delta}\iint_{Q_s}\int_0^1\Big|\frac{\partial\vec{A}}{\partial z}(x,t,\sigma u+(1-\sigma)v)d\sigma(u-v)\Big|^2dxdt
\end{aligned}$$

$$
\begin{aligned}
&+\delta \iint_{Q_s} \int_G \int_G |\xi-\zeta|^2 d\nu_{x,t}(\xi) d\mu_{x,t}(\zeta) dxdt \\
\le& \frac{1}{4\delta} M^2 \iint_{Q_s} (u-v)^2 dxdt + \delta \iint_{Q_s} \int_G \int_G |\xi-\zeta|^2 d\nu_{x,t}(\xi) d\mu_{x,t}(\zeta) dxdt. \qquad (3.53)
\end{aligned}
$$

Finally, (3.49) leads to

$$
\begin{aligned}
&\iint_{Q_s} \big(B(x,t,u)-B(x,t,v)\big)(u-v)dxdt \\
=& \iint_{Q_s} \int_0^1 \frac{\partial B}{\partial z}(x,t,\sigma u+(1-\sigma)v) d\sigma (u-v)^2 dxdt \\
\le& M \iint_{Q_s} (u-v)^2 dxdt. \qquad (3.54)
\end{aligned}
$$

Substituting (3.52)–(3.54) into (3.51), one obtains

$$
\begin{aligned}
&\int_\Omega (u(x,s)-v(x,s))^2 dx - \int_\Omega (u_0(x)-v_0(x))^2 dx \\
\le& \Big(\frac{1}{2\delta} M^2 + 2M\Big) \iint_{Q_s} (u(x,t)-v(x,t))^2 dxdt, \quad s \in [0,T],
\end{aligned}
$$

which leads to (3.50) by using the Gronwall inequality. □

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