Young, Old, Conservative and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing

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Abstract

We present a parsimonious and tractable general equilibrium model featuring a continuum of overlapping generations, as in Blanchard (1985). In addition, we assume that agents have standard utilities exhibiting constant relative risk aversion and can be born with differing risk aversions and endowments. We show that equilibrium asset prices are determined as if the economy was populated by a single representative agent with time-varying risk aversion that follows a stationary process. Moreover, the riskless rate is low and non-volatile. Therefore, despite its standard micro-foundation, our model is observationally similar to the external habit formation model of Campbell and Cochrane (1999), and is therefore successful at addressing a number of stylized facts about asset prices.

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1 Introduction

A significant body of research over the last two decades has focused on uncovering the link between variation in asset prices and fundamental macroeconomic risks. This has proven to be a challenging task. The baseline textbook consumption-smoothing model predicts too low equity premia, too high risk-free rates, and asset prices that are substantially less volatile than in the data, to name just a few of the widely documented failures. This has led many researchers to become pessimistic about the potential of rational consumption-based pricing models to explain observed asset valuations.

In this paper we argue that many of the failures of the standard consumption-based asset-pricing paradigm are linked to some of the simplifying assumptions behind it. Once we extend the standard textbook model of consumption smoothing in several simple and realistic dimensions, we find that the model can account reasonably well for many of the perceived failures of the consumption-based asset pricing model.

In particular, we take the following four main departures from the textbook model: a) Instead of assuming infinitely-lived agents, we acknowledge the fact that lives are finite and generations may not be altruistically linked through gifts or bequests; b) agents age, and their ability to produce declines with age; c) Moreover, they need not have the same endowments; some agents are workers, whereas some other agents run their own firms, and d) (aggregate) consumption and (aggregate) dividends are not equal — in particular, dividends are more volatile than consumption, but the two quantities are cointegrated over the long run.

All four extensions appear realistic and plausible. Furthermore, despite the vast diversity in the population that is introduced by the continuous arrival and departure of agents, their aging, and the differences in their preferences and endowments, we are able to obtain a fairly tractable model that addresses several asset-pricing puzzles in a transparent fashion.

The model is an extension of the perpetual youth model of Blanchard (1985). As in Blanchard (1985), agents arrive and die according to independent Poisson processes with constant intensity. In contrast to Blanchard (1985), however, firms are faced with a stochastic productivity process following a random walk. Hence, in contrast to Blanchard (1985), our

model is stochastic, so that we can analyze equity premia.

An advantage of assuming overlapping generations, as opposed to a single representative agent, is that the model can address the risk-free-rate puzzle. Since agents are faced with declining labor income over their life cycles, there is a constant pressure to save when they are young. The resulting increase in savings reduces the real interest rate and helps resolve the low risk-free rate puzzle.

A further departure from the standard model is that agents can have different risk aversions, and can derive their incomes from different sources. Less risk averse agents are more exposed to aggregate risk. Because of this, their relative importance in the wealth distribution increases (declines) in response to positive (negative) news. Thus, during good times, they are relatively more important in the economy and Sharpe ratios reflect their low risk-aversion levels. By contrast, in bad times, the economic importance of the less risk averse agents declines, and risky assets need to be held by the more risk averse agents, making Sharpe ratios reflect the high risk aversion of the latter group. Hence, even though each agent has constant relative risk aversion, assets are priced as if there existed a representative agent with time-varying risk aversion.

Importantly, because of the birth and death of agents, the wealth and the consumption distribution follow stationary processes. Similarly, risk premia, interest rates and valuation ratios are stationary. This stationarity can easily fail in models where agents are infinitely lived,[1] making it hard to compare these models to the data.

Another important feature of the model is the low volatility of the risk-less rates. This is driven by the fact that agents have different endowments and hence saving behaviors. In the model, the less risk averse agents are entrepreneurs rather than workers. Assuming that entrepreneurial income is on average higher but also more ephemeral than a worker's income, the less risk averse/entrepreneurial agents have a stronger propensity to save in order to smooth consumption intertemporally. On the other hand, the more risk averse/non-entrepreneurial agents have stronger precautionary motives for saving. As a consequence,

[1]See e.g. Dumas (1989) and Wang (1996).

even if the relative importance of the two types of agents changes in response to productivity shocks, aggregate savings (and the equilibrium interest rate) are unaffected. Hence, as in the Campbell and Cochrane (1999) model, almost all variation in discount rates is driven by changes in excess returns.

Additionally, by introducing an explicit labor-leisure choice and a production function with non-constant dividend and labor shares, we can reproduce the fact that dividends are more volatile than consumption, even though they are cointegrated over the long run. The higher volatility of dividends compared to consumption, along with the countercyclical variation in discount rates due to changing risk aversion, makes the volatility of the stock market high, which helps us obtain a reasonably high equity premium. Furthermore, because of the time-varying price of risk, the model can produce substantial predictability of excess returns.

Last, but not least, the model is quite tractable and easy to compute and analyze. Technically, the paper exploits the agents' ability to trade dynamically in order to derive the dynamics of the cross-sectional *consumption* distribution, an object that is easier to analyze than the *wealth* distribution.[2] Utilizing this insight, we are able to fully analyze properties of the equilibrium. Solving the model numerically amounts to computing the solution to a system of ordinary differential equations, a task that a modern computer can perform almost instantaneously. Moreover, there is no need for approximate solution concepts, whose accuracy is hard to assess.

The paper is related to various strands of the literature. There exists a vast literature on asset pricing that explains some of the stylized asset-pricing facts by utilizing habit formation. Constantinides (1990) and Abel (1990) were early contributions in this literature. Campbell and Cochrane (1999), in a highly influential paper, pursued the idea of external habit formation further. They succeeded in engineering a utility function exhibiting external habit formation that addresses several asset-pricing puzzles simultaneously.

[2]See also Basak and Cuoco (1998), who use a similar idea in the context of a model with limited-participation.

In the model that we propose, the state variable that governs time variation in asset prices resembles in many ways the "surplus" ratio of Campbell and Cochrane (1999). Hence, we are able to obtain a model that is observationally similar to Campbell and Cochrane (1999), but whose economic mechanisms and justification are substantially more standard. Therefore, besides being useful for understanding asset prices our model can also be used as a workhorse for performing welfare exercises, policy experiments, etc., inside a framework where asset prices are reasonably well matched, and the standard welfare theorems are not invalidated by substantial externalities.

Additionally, in our model dividends and consumption are different, yet co-integrated, so that the model can provide a laboratory to investigate the net present values of consumption, dividends, and labor income as separate quantities. These distinctions have attracted the attention of the recent asset-pricing literature.[3]

We also relate to Chan and Kogan (2002). Chan and Kogan (2002) present an interesting and insightful approach to obtaining a stationary wealth distribution in the presence of heterogeneity, based on habit-formation preferences. An advantage of their approach is that it allows for a continuum of risk aversions. However, it also produces substantial variability in interest rates, so that a substantial fraction of the equity premium is due to a pure term premium.

Several papers utilize variations in the cross-sectional wealth distribution due to some incompleteness to obtain implications for asset prices. This literature is vast and we do not attempt to summarize it. The papers that relate more closely to ours include Basak and Cuoco (1998), Guvenen (2005), Storesletten, Telmer, and Yaron (2007), and Gomes and Michaelides (2007). The first two of these papers assume infinitely lived agents, and the presence of limited participation allows the time variation of the wealth distribution to

[3]See, for instance, the work of Lettau and Ludvigson (2005), and Lustig and Van Nieuwerburgh (2007). We further relate to Santos and Veronesi (2006) and Menzly, Santos, and Veronesi (2004), since both these papers produce a dividend process that is not identical to consumption in the short run, but is cointegrated over the long run. The important difference is that in our paper this share process arises endogenously and *jointly* with the time variation in discount rates.

affect returns. A common tendency of models with infinitely lived agents is that wealth eventually concentrates in the hands of stock market participants. Even though ours is not a model of limited stock market participation, the presence of differing risk aversions has observationally similar implications. More importantly, the assumption of overlapping generations implies that all agents start and end life with zero wealth, so that the equity premium does not reflect, asymptotically, only the risk aversion of one group. Furthermore, an improvement over Guvenen (2005) is that our interest-rate volatility is very low and our consumption process exhibits growth and is practically unpredictable. Storesletten, Telmer, and Yaron (2007) and Gomes and Michaelides (2007) study overlapping generations models and introduce frictions. Storesletten, Telmer, and Yaron (2007) study changes in the cross-sectional variation of consumption shocks as Constantinides and Duffie (1996). Gomes and Michaelides (2007) analyze a rich setup (costly participation, heterogeneity in both preferences and income, etc.) and focus on understanding individual portfolio holdings in general equilibrium. However, both Storesletten, Telmer, and Yaron (2007) and Gomes and Michaelides (2007) impose effectively an *exogenous* stock market volatility, by adopting a setup that is closer in spirit to Cox, Ingersoll, and Ross (1985) than to Lucas (1978). In such a framework the *quantity* rather than the *price* of capital absorbs all economic shocks, since Tobin's q equals 1. Hence, the volatility in stock valuations results from exogenous assumptions on stochastic depreciation, and the volatility of aggregate output is driven to a large extent by changes in the capital stock, rather than total factor productivity. As Storesletten, Telmer, and Yaron (2007) admit, "solving the analogous endowment economy is substantially more difficult". The reason is that, in an endowment economy of the Lucas (1978) kind, volatility in asset prices is *endogenous*, since it is the price rather than the quantity of capital that absorbs all shocks. Because volatility is both challenging and central for many other asset pricing moments (such as the equity premium, the predictability of returns, etc.), we believe that our framework allows us to address a broader set of asset-pricing puzzles compared to previous literature.

Moreover, because of the analytical tractability of our framework, we can inspect more

closely *how* heterogeneity affects asset prices. In this sense our work is complementary to quantitative exercises, that feature a richer setup, but at the cost of making the mechanisms less transparent.

There is a vast literature on overlapping-generations models. A partial listing of interesting applications of OLG frameworks to asset pricing include Abel (2003), Constantinides, Donaldson, and Mehra (2002), and Heaton and Lucas (2000b). Most models in the OLG tradition share the feature that the smallest time periods of the model correspond to decades. The advantage of using a Blanchard (1985) framework is that the model produces implications for any time interval of interest. Given that most empirical regressions are run with monthly, quarterly, or yearly data, this makes it easier to relate the model to the empirical asset-pricing literature.

Finally, there is a large literature on entrepreneurship and inequality in general equilibrium, which we do not attempt to summarize here. This literature has a fundamentally different focus than this paper. The goal of this literature is to understand decisions of entrepreneurs and their implications for inequality. Typically, these papers abstract from aggregate uncertainty so as to preserve tractability. Because of our focus on asset pricing, we cannot do the same. Instead, we abstract from many realistic aspects of entrepreneurship and focus on the asset pricing implications of the model. For our purposes, the most important aspect of entrepreneurship is the fact that entrepreneurs save more than non-entrepreneurs, and that entrepreneurial income is more correlated with aggregate output and the stock market. Both of these facts are consistent with the data[4]and reproduced by the model.

Section 2 describes the model. Section 3 presents the solution of the model. Section 4 presents a qualitative discussion and Section 5 contains quantitative implications. Section 6 contains a brief discussion of the model's implications beyond asset pricing. Section 7 concludes. All proofs are contained in the appendix.

[4]For evidence on the first fact see Gentry and Hubbard (2002) and for evidence on the latter see e.g. Heaton and Lucas (2000a).

2 Model

2.1 Agents' Preferences and Endowments

There is a continuum of agents whose mass we normalize to 1. Existing agents face a constant hazard rate of death $\pi > 0$ throughout their lives. Furthermore, newly born agents also arrive at a rate of π per unit of time, so that the population remains constant. These demographic assumptions are identical to Blanchard (1985) and are key for the tractability of all the aggregation results.

As is standard in the literature, we furthermore assume that agents have constant relative risk aversion. A key departure from prevailing representative-agent approaches is that we explicitly allow for the possibility that agents have heterogenous preferences. The most parsimonious way to introduce heterogeneity is to follow Dumas (1989) and Wang (1996) and assume the presence of two types of agents, which we label as "type-A" and "type-B" agents.

Letting s denote the time of an agent's birth and τ the (stochastic) time of her death, type-A agents have mass υ and preferences of the form

$$E_s \int_s^\tau e^{-\rho^A (t-s)} \frac{\left(c^A_{t,s}\right)^{1-\gamma^A}}{1-\gamma^A} dt. \tag{1}$$

$\rho^A > 0$ is the subjective discount rate and $\gamma^A > 0$ the relative risk aversion of type-A agents. Throughout we follow the notational convention that $c^i_{t_1,t_2}$ denotes the time t_1 consumption by an agent of type $i \in \{A, B\}$ who was born at time $t_2 \leq t_1$. By integrating over the (exponential) distribution of the time of death in (1), the agent's problem amounts to maximizing[5]

$$E_s \int_s^\infty e^{-(\rho^A+\pi)(t-s)} \frac{\left(c^A_{t,s}\right)^{1-\gamma^A}}{1-\gamma^A} dt. \tag{2}$$

Type-A agents come into life endowed with units of non-tradeable (human) capital. We postpone a more detailed discussion of a type-A agent's endowment until the next section, when we describe the production technology.

[5]This is a standard property of the Blanchard-Yaari model. See Blanchard (1985).

The second type of agents (type-B agents) have mass $1-\upsilon$. They arrive in life with an endowment of hours, which they can supply in markets in exchange for wage income. We follow Blanchard (1985) and assume that the agents' endowments of hours decline exponentially over the life-cycle at the rate χ. Blanchard (1985) argues that this simple assumption captures the idea that agents retire,[6] so that their income over the life cycle is downward-sloping.[7]

Type-B agents maximize an objective of the form

$$E_s \int_s^{\infty} e^{-(\rho^B+\pi)(t-s)} \frac{\left(\left(c_{t,s}^B\right)^{\psi}\left(\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon}e^{-\chi(t-s)}-h_{t,s}^B\right)^{1-\psi}\right)^{1-\gamma^B}}{\psi\left(1-\gamma^B\right)}dt. \tag{3}$$

$\rho^B > 0$ is the subjective discount rate and $\gamma^B > 0$ the relative risk aversion of type-B agents. The constant $\psi \in (0,1]$ controls how much type-B agents care about leisure, and $h_{t,s}^B$ refers to the time-t hours that are supplied by an agent who was born at time s. The term $\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon}e^{-\chi(t-s)}$ gives the time t endowment of hours for an agent who was born at time s. The normalizing constant $\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon}$ that multiplies a type-B agent's endowment implies that the aggregate endowment of hours in the economy is unity, since

$$(1-\upsilon)\int_{-\infty}^{t} \pi e^{-\pi(t-s)}\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon}e^{-\chi(t-s)}ds = 1. \tag{4}$$

Note that Equation (4) accounts for the mass of type-B agents $(1-\upsilon)$ and their age distribution by the term $\pi e^{-\pi(t-s)}$ inside the integral.

[6] Admittedly, even though the Blanchard (1985) assumption captures the idea that agents save for retirement, it cannot capture the full richness of realistic life cycle income profiles, that tend to be upward sloping in the early stages of the life cycle. However, it is reasonable to conjecture that agents are also borrowing constrained in the early stages of their life cycles, so that they do not accumulate assets and hence their effect on asset prices is likely to be limited.

[7] We remark here that our results would be unchanged if we assumed that agents have a constant endowment of hours that they lose for ever at some random exponentially distributed time that arrives with intensity χ. One could interpret such a situation as a health shock that leads to retirement. If we interpret the model in this way, we need to also assume the existence of health insurance markets, in order to preserve tractability. In those markets agents would be able to enter contracts that deliver payoffs contingent on the arrival of the health shock.

Throughout, we shall assume that $\gamma^A \leq \gamma^B$, so that type-B agents are at least as risk averse as type-A agents.

2.2 Technology

Firms come in two varieties: publicly traded and privately held. We start by describing the first kind and then turn to the second kind.

The publicly traded firms are competitive and identical to each other, and own a total (physical) capital stock equal to K_t^{pub}. Assuming constant returns to scale, we can therefore proceed as if there was a single "representative" firm that owns K_t^{pub}units of the (physical) capital stock. This firm produces output Y_t^{pub} by utilizing a technology of the form

$$Y_t^{\text{pub}} = Z_t K_t^{\text{pub}} f\left(\frac{H_t^{\text{pub}}}{K_t^{\text{pub}}}\right), \tag{5}$$

where Z_t follows a geometric Brownian motion

$$\frac{dZ_t}{Z_t} = \mu_Z dt + \sigma_Z dB_t$$

for two positive constants μ_Z and σ_Z. H_t^{pub} denotes the aggregate hours worked at the representative public firm.

The function $f(\cdot)$ in (5) is an increasing and concave function of the hours-to-capital ratio $\eta_t \equiv H_t^{\text{pub}}/K_t^{\text{pub}}$. In particular, we assume that $f(\eta_t)$ solves the following ordinary differential equation:

$$f'(\eta) = \frac{\alpha(\eta) f(\eta)}{\eta},\ f(0) = 0 \tag{6}$$

and that $\alpha(\eta)$ is a continuous function satisfying

$$\alpha(\eta) \in (0,1),\ \alpha'(\eta) \leq 0. \tag{7}$$

First, note that $f'(\eta) > 0$, given the above assumptions. Second, differentiating both sides of (6) and using (7), it follows that $f''(\eta) < 0$. Furthermore, in the special case where $\alpha(\eta)$ is constant and equal to α, the resulting solution to (6) is the familiar Cobb-Douglas

production function $f(\eta) = \eta^{\alpha}$. When $\alpha(\eta)$ is chosen as $\alpha(\eta) = \frac{(1-b)\eta^{-\nu}}{(1-b)\eta^{-\nu}+b}$ for some $\nu > 0$ and some $0 < b < 1$, then $f(\eta)$ specializes to the CES production function. As in Lucas (1978), we assume that $K_t^{\text{pub}} = K^{\text{pub}}$ remains constant throughout, i.e., the (physical) capital stock does not depreciate. Finally, publicly traded firms are fully equity financed with the total supply of their equity normalized to 1.

We now turn to privately held firms. We think of these as entrepreneurial firms whose existence at time t depends on some crucial skill of an agent who is alive at time t. Once a type-A agent is born, an entrepreneurial firm is born, and when a type-A agent dies, her firm perishes as well. In the baseline model we simply assume that type A agents are simultaneously entrepreneurial and have lower risk aversion than type B agents. We revisit this issue in section 6.2, and show how to obtain the link between entrepreneurship and lower risk aversion in an extended model with uncertain and idiosyncratic entrepreneurial skills.

Given these simplifying assumptions, it is possible to think of the present setup as a situation where type-A agents arrive in life endowed with units of human/entrepreneurial "capital"stock $K_{s,s}^{\text{priv}}$. Following a popular approach in the literature[8], we think of entrepreneurial capital as a productive factor in a broad sense, i.e., as an idea or managerial talent, that can be used to produce output once combined with labor. Purely for parsimony, we assume that entrepreneurial capital gives entrepreneurs access to a technology that produces output (once combined with labor) according to a production technology that is similar to (5), i.e.,[9]

$$Y_{t,s}^{\text{priv}} = Z_t K_{t,s}^{\text{priv}} f\left(\frac{h_{t,s}^{A}}{K_{t,s}^{\text{priv}}}\right), \tag{8}$$

where $h_{t,s}^{A}$ are the time-t hours that are *demanded* by the firm of a type-A agent who was born at time s. To capture the idea that the ongoing arrival of new entrepreneurial firms may make the ideas and skills of previous entrepreneurs partially obsolete, we allow the capital stock of entrepreneurs to depreciate with their age at the rate $\delta \geq 0$, so that $K_{t,s}^{\text{priv}} = e^{-\delta(t-s)} K_{s,s}^{\text{priv}}$.

[8] See, e.g., Quadrini (2000).

[9] One possible interpretation of equation (8) is that entrepreneurs can run their firms more efficiently than public firms, and hence can use their skill to economize on physical capital. In that interpretation physical and entrepreneurial capital are (perfect) substitutes.

Type A agents cannot trade the units of the (human) capital stock that they are endowed with. For them, the output Y_t^{priv} (net of the wages that they pay to the workers that they hire) corresponds to their non-traded income. All generations of type-A agents come endowed with the same (human) capital stock, and we will simplify notation by writing $K_{s,s}^{\text{priv}} = K^{\text{priv}}$.

Finally, as a matter of normalization, we assume that $K^{\text{pub}} \in (0,1)$ and that $K^{\text{priv}} = \frac{1}{\upsilon}\frac{\pi+\delta}{\pi}\left(1-K^{\text{pub}}\right)$. Given these assumptions, the total capital stock K_t in the economy is constant and given by

$$K_t = \upsilon K^{\text{priv}} \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} ds + K^{\text{pub}} = \upsilon K^{\text{priv}} \frac{\pi}{\pi+\delta} + K^{\text{pub}} = 1. \tag{9}$$

The integral in equation (9) adds up the capital stock that belongs to all surviving generations of type-A agents, taking into account the mass υ of these agents and their age distribution. Given that the capital stock adds up to 1, K^{pub} can be interpreted as the fraction of the capital stock that belongs to the publicly traded firm.

2.3 Budget Constraints

A type-B agent who supplies $h_{t,s}^B$ hours of labor at time t earns a labor income of $w_t h_{t,s}^B$, where w_t is the prevailing wage at time t. Similarly, a type-A agent receives dividends from her firm that are equal to $Y_{t,s}^{\text{priv}} - w_t h_{t,s}^A$. Hence, the time-$t$ income of an agent of type $i = \{A, B\}$ who was born at time s is given by

$$y_{t,s}^i = \begin{cases} Y_{t,s}^{\text{priv}} - w_t h_{t,s}^A, & \text{if } \ i = A; \\ w_t h_{t,s}^B, & \text{if } \ i = B. \end{cases} \tag{10}$$

Both agents can also trade in a risk-less bond, a stock, and competitively priced annuities as in Blanchard (1985). The rate of return on bonds is given by r_t. The stock is a claim that delivers a dividend flow given by $D_t \equiv Y_t^{\text{pub}} - w_t H_t^{\text{pub}}$. It is reasonable to conjecture that the stock-price process follows a diffusion:

$$dS_t = (\mu_t S_t - D_t)dt + \sigma_t S_t dB_t \tag{11}$$

for some processes μ_t and σ_t. The processes for w_t, r_t, μ_t, and σ_t will be jointly determined later so that markets clear. Finally, competition amongst competitive life insurers will drive the annuity income per dollar annuitized toward the actuarially fair flow of π per unit of time.[10]

For future reference, it is convenient to define the stochastic discount factor process as

$$\frac{d\xi_t}{\xi_t} \equiv -r_t dt - \kappa_t dB_t, \tag{12}$$

where κ_t is the Sharpe ratio in the market defined as $\kappa_t = \frac{\mu_t - r_t}{\sigma_t}$. Given the assumptions, the financial wealth $W^j_{t,s}$ of agent $i = \{A, B\}$ evolves as

$$dW^i_{t,s} = \left(r_t W^i_{t,s} + \pi W^i_{t,s} - c^i_{t,s} + \theta^i_{t,s}(\mu_t - r_t) + y^i_{t,s}\right) dt + \theta^i_{t,s}\sigma_t dB_t, \tag{13}$$

where $\theta^i_{t,s}$ denotes the dollar investment in publicly traded stocks. Equation (13) is a standard dynamic budget constraint. The term $\pi W^i_{t,s}$ captures the fact that the agent has no bequest motive and hence will choose to annuitize her entire wealth[11]. Letting W_t denote aggregate wealth, insurance companies collect πW_t per unit of time from agents who die, and hence can deliver payments equal to πW_t to the survivors. As a result of that, Blanchard (1985) shows that insurance companies break even at each point in time and the competitive annuity price is identical to the actuarially fair price. For the rest of the paper, we will be concerned with clearing the remaining markets.

2.4 Markets and Equilibrium

There are four markets that must clear in equilibrium: 1) the labor market; 2) the current-consumption-good market; 3) the bond market, where agents trade a zero net supply bond,

[10] See Blanchard (1985) for details.

[11] To be more specific, annuities work as follows in this context. The agent signs an instantaneous contract that delivers competitive insurers a fraction $\eta^i_{t,s}$ of her wealth upon death in exchange for an income of $\hat{\pi}_t \eta^i_{t,s} W_{t,s}$ while the agent is alive. Since the agent has no bequest motives, and annuities are fairly priced (i.e., $\hat{\pi}_t = \pi$), it follows that $\eta^i_{t,s} = 1$. For details see Blanchard (1985).

and 4) the stock market, where agents trade claims to the dividend of the publicly traded company.

The definition of equilibrium is standard:

Definition 1 *An equilibrium is defined as a set of progressively measurable processes* $\{c^i_{t,s}, \theta^i_{t,s}, h^i_{t,s}\}$ *for* $i \in \{A, B\}$ *and a set of progressively measurable processes for the rate of return in the bond market* (r_t), *wages* (w_t) *and an appropriate stock market process of the form (11) with progressively measurable coefficients* μ_t, σ_t *such that:*

1. *Given the process for* $\{r_t, w_t, \mu_t, \sigma_t\}$, *for all* s *and* t *with* $t \geq s$, *the processes* $\{c^i_{t,s}, \theta^i_{t,s}, h^i_{t,s}\}$ *for* $i = A, B$ *maximize (2) (objective [3] respectively) subject to (13), the initial condition* $W^i_{s,s} = 0$ *and the transversality condition* $E_s \lim_{t\to\infty} e^{-\pi t}\xi_t W^i_{t,s} = 0$ *for all* s.

2. *Given* w_t, *public firms*[12] *choose hours so as to maximize profits:*

$$H^{pub}_t = \arg\max_{H^{pub}_t} D_t \tag{14}$$

3. *Given* $c^i_{t,s}, h^i_{t,s}, \theta^i_{t,s}$ *for* $i \in \{A, B\}$, *markets for goods and labor clear, i.e.,*

$$(1-\upsilon)\int_{-\infty}^{t} \pi e^{-\pi(t-s)} h^B_{t,s} ds = H^{pub}_t + \upsilon \int_{-\infty}^{t} \pi e^{-\pi(t-s)} h^A_{t,s} ds \tag{15}$$

$$\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left(\upsilon c^A_{t,s} + (1-\upsilon) c^B_{t,s}\right) ds = Y^{pub}_t + \upsilon \int_{-\infty}^{t} \pi e^{-\pi(t-s)} Y^{priv}_{t,s} ds \tag{16}$$

and markets for stocks and bonds clear, as well:

$$\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left(\upsilon \theta^A_{t,s} + (1-\upsilon)\theta^B_{t,s}\right) ds = S_t \tag{17}$$

$$\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left(\upsilon \left(W^A_{t,s} - \theta^A_{t,s}\right) + (1-\upsilon)\left(W^B_{t,s} - \theta^B_{t,s}\right)\right) ds = 0. \tag{18}$$

[12]Note that it is sufficient to require that public firms maximize profits. The requirement that private firms should also maximize profits is implied by the first part of the definition. The reason is that type-A agents choose the hours demanded by their firms in an expected-utility maximizing way. As is shown in the next section, this is equivalent to maximizing the flow of profits at each state and date.

Equation (15) states that the aggregate hours supplied by type-B agents alive at time t equal the total hours demanded by publicly traded and privately held firms. Equations (16), (17), and (18) capture the analogous requirements for the goods market, the stock market, and the bond market.

3 Solution

In this section we construct an equilibrium. We start by letting $Y_t = Y_t^{\text{pub}} + \int_{-\infty}^{t} \pi e^{-\pi(t-s)} Y_{t,s}^{\text{priv}} ds$ denote the aggregate output in the economy, and X_t denote the consumption share of type-A agents, namely

$$X_t \equiv \frac{\upsilon \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c_{t,s}^{A} ds}{Y_t}. \tag{19}$$

Since the consumption of both types of agents is non-negative, the goods-market clearing condition (16) implies that $X_t \in [0,1]$. In the remainder of this section we construct an equilibrium with the following properties: a) X_t is a Markov process b) $r_t, \mu_t, \sigma_t, \kappa_t$ are functions of X_t exclusively, and w_t has the form $w_t = Z_t \omega(X_t)$ for an appropriate function ω that we determine explicitly. In practical terms, this implies that a single variable, namely X_t is sufficient to characterize the equilibrium interest rate, expected stock market returns and volatility.

3.1 Labor Markets

To establish the claims above, we start by examining the labor market. Labor supply by type-B agents is determined by the familiar condition that the marginal rate of substitution between leisure and consumption should be equal to the real wage. Letting $u(c_{t,s}^{B}, h_{t,s}^{B})$ denote the utility of a type-B agent, we obtain

$$-\frac{u_h\left(c_{t,s}^{B}, h_{t,s}^{B}\right)}{u_c\left(c_{t,s}^{B}, h_{t,s}^{B}\right)} = w_t. \tag{20}$$

Using the functional-form of $u\left(c_{t,s}^{B}, h_{t,s}^{B}\right)$ and (20) gives the following relationship between hours, consumption, and wages:

$$h_{t,s}^{B} = \frac{1}{1-\upsilon}\frac{\pi+\chi}{\pi}e^{-\chi(t-s)} - \frac{(1-\psi)}{\psi}\frac{c_{t,s}^{B}}{w_t} \qquad \text{for } i \in \{A, B\}. \tag{21}$$

Letting H_t denote the aggregate hours supplied in the economy and using (21) along with (19) gives

$$H_t = \int_{-\infty}^{t} \pi e^{-\pi(t-s)}\left(1-\upsilon\right) h_{t,s}^{B} ds = 1 - \frac{Y_t}{w_t}\frac{(1-\psi)}{\psi}\left(1-X_t\right). \tag{22}$$

This expression gives the aggregate labor-supply relation implied by the model. To clear the labor market, we turn attention to the aggregate labor demand. Whether privately held or publicly traded, a firm can maximize its profits state by state by simply setting the marginal product of labor equal to the real wage. For both publicly traded and privately held firms this leads to the first order conditions

$$Z_t f'\left(\frac{H_t^{\text{pub}}}{K^{\text{pub}}}\right) = Z_t f'\left(\frac{h_{t,s}^{A}}{K_{t,s}^{A}}\right) = w_t. \tag{23}$$

An implication of (23) is that all firms in the economy will have the same hours-to-capital ratio, which in turn will be equal to the aggregate hours-to-capital ratio. Letting that ratio be denoted by η_t^{aggr}, it follows that in equilibrium $\eta_t^{\text{aggr}} = \frac{H_t^{\text{pub}}}{K^{\text{pub}}} = \frac{h_{t,s}^{A}}{K_{t,s}^{A}} = \frac{H_t}{K_t} = H_t$ since $K_t = 1$. Using these observations to compute aggregate output gives

$$\begin{aligned} Y_t &= Y_t^{\text{pub}} + \upsilon \int_{-\infty}^{t} \pi e^{-\pi(t-s)} Y_{t,s}^{\text{priv}} ds = \\ &= Z_t K^{\text{pub}} f\left(\frac{H_t^{\text{pub}}}{K^{\text{pub}}}\right) + \upsilon Z_t K^{\text{priv}} \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} f\left(\frac{h_{t,s}^{A}}{K_{t,s}^{A}}\right) ds \\ &= Z_t f(H_t)\left[K^{\text{pub}} + \upsilon K^{\text{priv}} \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} ds\right] = Z_t f(H_t) K_t = Z_t f(H_t). \end{aligned} \tag{24}$$

Using $Y_t = Z_t f(H_t)$ together with (6), (5), and (23) and noting that $f'\left(H_t^{\text{pub}}/K^{\text{pub}}\right) = f'(H_t)$ leads to

$$\frac{Y_t}{w_t} = \frac{H_t}{\alpha(H_t)}. \tag{25}$$

Using (25) inside (22) results in

$$H_t = 1 - \frac{H_t}{\alpha(H_t)} \frac{(1-\psi)}{\psi} (1 - X_t) . \tag{26}$$

Given a value of X_t, equation (26) determines the equilibrium quantity of hours implied by the model. We shall therefore write $H_t = \mathcal{H}(X_t)$ to denote the dependence of hours on X_t. Furthermore, equation (23) implies that the equilibrium wage can be written in the form $w_t = Z_t f'(\mathcal{H}(X_t))$. It will be useful to define $\omega(X_t) \equiv f'(\mathcal{H}(X_t))$, so that the resulting equilibrium wage can be expressed as $w_t = Z_t \omega(X_t)$ as we conjectured at the beginning of section 3.

3.2 Intertemporal Consumption Allocations

To study agents' intertemporal consumption decisions, we start by assuming that there exists a stochastic discount factor ξ_t. Given that agents can trade dynamically in stocks and bonds and there is a single source of uncertainty, agent A's optimal consumption choices satisfy the intertemporal first order condition[13]

$$e^{-\left(\pi+\rho^A\right)(t-s)} \left(\frac{c^A_{t,s}}{c^A_{s,s}} \right)^{-\gamma^A} = e^{-\pi(t-s)} \frac{\xi_t}{\xi_s}. \tag{27}$$

Equation (27) captures the intertemporal aspect of agent A's problem. Roughly speaking, it states that the marginal benefit of an additional unit of consumption in a given state (as measured by the marginal utility of consumption) should equal the "cost" of a unit of consumption in that state. In turn this "cost" is measured by the product of the stochastic discount factor and the probability that the consumer will live until time t (namely $e^{-\pi(t-s)}$). Agent B's intertemporal first order condition is given by

$$e^{-\left(\pi+\rho^B\right)(t-s)} \left(\frac{\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon} e^{-\chi(t-s)} - h^B_{t,s}}{\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon} - h^B_{s,s}} \right)^{\left(1-\gamma^B\right)(1-\psi)} \left(\frac{c^B_{t,s}}{c^B_{s,s}} \right)^{\psi\left(1-\gamma^B\right)-1} = e^{-\pi(t-s)} \frac{\xi_t}{\xi_s}. \tag{28}$$

[13] For details see Karatzas and Shreve (1998), Chapter 3.

Equation (28) is similar to equation (27), modified by the fact that the marginal utility of consumption depends also on leisure when $\psi < 1$. Re-arranging (28) leads to

$$\frac{c^B_{t,s}}{c^B_{s,s}} = e^{-\frac{\rho^B}{1-\psi(1-\gamma^B)}(t-s)} \left(\frac{\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon}e^{-\chi(t-s)} - h^B_{t,s}}{\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon} - h^B_{s,s}} \right)^{-\frac{(1-\psi)\left(\gamma^B-1\right)}{1-\psi(1-\gamma^B)}} \left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{1-\psi(1-\gamma^B)}}. \tag{29}$$

We observe that, since (21) has to hold at all dates and states, it implies the following relation between consumption, hours, and wages between two different points in time:

$$\frac{\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon}e^{-\chi(t-s)} - h^B_{t,s}}{\frac{\pi+\chi}{\pi}\frac{1}{1-\upsilon} - h^B_{s,s}} = \frac{c^B_{t,s}}{c^B_{s,s}}\frac{w_s}{w_t}. \tag{30}$$

Combining (29) with (30) and rearranging leads to

$$\frac{c^B_{t,s}}{c^B_{s,s}} = e^{-\frac{\rho^B}{\gamma^B}(t-s)} \left(\frac{w_t}{w_s}\right)^{\frac{(1-\psi)\left(\gamma^B-1\right)}{\gamma^B}} \left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma^B}}. \tag{31}$$

Similarly, rearranging (27) leads to

$$\frac{c^A_{t,s}}{c^A_{s,s}} = e^{-\frac{\rho^A}{\gamma^A}(t-s)} \left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma^A}}. \tag{32}$$

Given $c^A_{s,s}, c^B_{s,s}$ equations (31) and (32) give the entire stochastic process of any agent's consumption profile as a function of wages and the stochastic discount factor. In order to determine the initial consumption $(c^i_{s,s})$ for $i \in \{A, B\}$ we use the definition of income $y^i_{t,s}$ in equation (10) along with the inter-temporal budget constraint to obtain

$$E_s \int_s^\infty e^{-\pi(t-s)} c^i_{t,s} \left(\frac{\xi_t}{\xi_s}\right) dt = E_s \int_s^\infty e^{-\pi(t-s)} y^i_{t,s} \left(\frac{\xi_t}{\xi_s}\right) dt. \tag{33}$$

This states that the consumer's net present value of consumption over the life cycle should equal the net present value of her non-traded income (since she is born with zero financial wealth).

Letting Φ^A_s denote the time s net present value of non-traded income for a newly-born type-A agent, substituting (10) into (33), using (8), (6) and (23) along with $K^{\text{priv}}_{t,s} = K^{\text{priv}} e^{-\delta(t-s)}$ and recalling that $K_t = 1$ we obtain

$$\Phi^A_s \equiv E_s \int_s^\infty e^{-\pi(t-s)} y^i_{t,s} \left(\frac{\xi_t}{\xi_s}\right) dt = K^{\text{priv}} E_s \int_s^\infty e^{-(\pi+\delta)(t-s)} Y_t \left(1 - \alpha(H_t)\right) \left(\frac{\xi_t}{\xi_s}\right) dt\,. \tag{34}$$

Similarly, for agents of type B using (10), and (21) inside the right hand side of (33) and rearranging gives

$$E_s \int_s^\infty e^{-\pi(t-s)} c^B_{t,s} \left(\frac{\xi_t}{\xi_s}\right) dt = \psi \frac{1}{1-\upsilon} \frac{\pi+\chi}{\pi} E_s \int_s^\infty e^{-(\pi+\chi)(t-s)} w_t \left(\frac{\xi_t}{\xi_s}\right) dt. \tag{35}$$

In light of this equation, we shall define Φ^B_s (the net present value of a type B agent's income) as:

$$\Phi^B_s \equiv \psi \frac{1}{1-\upsilon} \frac{\pi+\chi}{\pi} E_s \int_s^\infty e^{-(\pi+\chi)(t-s)} w_t \left(\frac{\xi_t}{\xi_s}\right) dt. \tag{36}$$

An implication of our conjecture that X_t is a Markov process and that r_t and κ_t are functions of X_t is the following Lemma.

Lemma 1 *Assuming that X_t is a Markov process and that r_t and κ_t are functions of X_t , there exist four functions $\phi^i(X_t)$ and $\beta^i(X_t)$ for $i \in \{A, B\}$, such that $\Phi^i_s = \phi^i(X_s) Y_s$ and $c^i_{s,s} = \beta^i(X_s) Y_s$.*

The next sub-section derives the dynamics of X_t, r_t, κ_t and confirms the conjecture that X_t is Markov, along with the conjecture that r_t, κ_t are functions of X_t.

3.3 Dynamics of the Stochastic-Discount Factor and the Consumption Share

We start by determining the processes for the drift $\mu_X(t)$ and diffusion coefficients $\sigma_X(t)$ of

$$dX_t = \mu_X dt + \sigma_X dB_t. \tag{37}$$

To simplify notation we will use the short-hand notation (μ_X, σ_X) instead of $(\mu_X(t), \sigma_X(t))$. To determine the dynamics of X_t, we start by defining the function $g(X_t)$ as

$$g(X_t) \equiv \frac{Y_t}{Z_t} = f(\mathcal{H}(X_t)). \tag{38}$$

Using (24), (38), and Ito's Lemma implies that $dY_t/Y_t = \mu_Y dt + \sigma_Y dB_t$, where

$$\mu_Y = \mu_Z + \frac{g'}{g}(\mu_X + \sigma_X \sigma_Z) + \frac{\sigma_X^2}{2} \frac{g''}{g} \tag{39}$$

$$\sigma_Y = \sigma_Z + \frac{g'}{g} \sigma_X. \tag{40}$$

Combining (32), Lemma 1, the definition of g in equation (38), and the definition of X_t in equation (19) leads to

$$X_t Y_t = \upsilon \int_{-\infty}^{t} \pi e^{-\left(\pi+\frac{\rho^A}{\gamma^A}\right)(t-s)} \beta_s^A Z_s g_s \left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma^A}} ds, \tag{41}$$

where we have used the shorthand notation $g_s = g(X_s), \beta_s^A = \beta^A(X_s)$. Applying Ito's Lemma on both sides of (41), using (12), and matching the resulting diffusion coefficients on the left- and the right-hand sides implies that

$$\frac{\sigma_X}{X_t} + \sigma_Y = \frac{\kappa_t}{\gamma^A}. \tag{42}$$

Similarly, matching the drift coefficients on both sides gives

$$\mu_X + \mu_Y X_t + \sigma_Y \sigma_X = X_t \left[\frac{r_t - \rho^A}{\gamma^A} + \frac{\kappa_t^2}{2}\frac{\gamma^A + 1}{(\gamma^A)^2} - \pi\right] + \upsilon\pi\beta_t^A, \tag{43}$$

To solve for μ_X and σ_X from equations (42) and (43) we need to obtain expressions for r_t and κ_t, which we do by using the goods-market clearing condition (16). Specifically, combining (16) with (31) and (32) gives

$$\begin{aligned} Y_t &= \int_{-\infty}^{t} \pi e^{-\left(\pi+\frac{\rho^A}{\gamma^A}\right)(t-s)} \upsilon \beta_s^A Z_s g_s \left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma^A}} ds + \\ &\quad \int_{-\infty}^{t} \pi e^{-\left(\pi+\frac{\rho^B}{\gamma^B}\right)(t-s)} (1-\upsilon) \beta_s^B Z_s g_s \left(\frac{Z_t \omega_t}{Z_s \omega_s}\right)^{\frac{(1-\psi)\left(\gamma^B-1\right)}{\gamma^B}} \left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma^B}} ds. \end{aligned} \tag{44}$$

Once again, applying Ito's Lemma to both sides of (44) and matching diffusion terms on the left- and the right-hand sides yield

$$\sigma_Y = X_t \frac{\kappa_t}{\gamma^A} + (1 - X_t)\left[\frac{\kappa_t}{\gamma^B} + \frac{(1-\psi)\left(\gamma^B - 1\right)}{\gamma^B}\left(\frac{\omega'}{\omega}\sigma_X + \sigma_Z\right)\right]. \tag{45}$$

Similarly, by matching drift coefficients we obtain

$$\begin{aligned} \mu_Y &= \sum_{i\in\{A,B\}} \upsilon^i \pi \beta_t^i + X_t \left[\frac{r_t - \rho^A}{\gamma^A} + \frac{\kappa_t^2}{2}\frac{\gamma^A + 1}{(\gamma^A)^2} - \pi\right] + \\ &\quad (1 - X_t)\left[\mathcal{D}\left(w^{\frac{(1-\psi)\left(\gamma^B-1\right)}{\gamma^B}} \xi^{-\frac{1}{\gamma^B}}\right) - \left(\pi + \frac{\rho^B}{\gamma^B}\right)\right], \end{aligned} \tag{46}$$

where we have used the shorthand notation $\upsilon^A = \upsilon$, $\upsilon^B = 1 - \upsilon$, and

$$\begin{aligned}\mathcal{D}\left(w^a \xi^b\right) &\equiv a\left(\mu_Z + \frac{\sigma_Z^2}{2}(a-1) + \frac{\omega'}{\omega}\left(\mu_X + a\sigma_Z\sigma_X - \kappa_t b \sigma_X\right) - b\kappa_t\sigma_Z\right) \\ &\quad + \frac{\sigma_X^2}{2}\left(a(a-1)\left(\frac{\omega'}{\omega}\right)^2 + a\frac{\omega''}{\omega}\right) - b\left(r_t + \frac{\kappa_t^2}{2}(1-b)\right).\end{aligned} \tag{47}$$

Fixing a value of X_t, equations (42) and (45) form a linear system in σ_X and κ_t that can be solved explicitly. This yields σ_X and κ_t as functions of X_t. Having obtained σ_X and κ_t, equations (43) and (46) also form a linear system in r_t and μ_X that can be solved explicitly, yielding r_t and μ_X as functions of X_t, as conjectured at the beginning of the section. Since both μ_X and σ_X are functions of X_t, the consumption-share process X_t is a Markov process, verifying the guess at the beginning of section 3.

The last step in the construction of the equilibrium stochastic discount factor is the explicit determination of the functions $\phi^i(X_t)$ and $\beta^i(X_t)$. The following Lemma shows how to obtain these functions.

Lemma 2 *Let $\sigma_X(X_t), \kappa(X_t), \mu_X(X_t)$, and $r(X_t)$ denote the solution to the linear system of equations (42), (43), (45), (46). Then the function $\phi^A(X_t)$ and $\phi^B(X_t)$ are the solutions to the differential equations*

$$0 = \frac{\sigma_X^2}{2}\left(\phi^A\right)'' + \left(\phi^A\right)'\left(\mu_X + \sigma_X(\sigma_Y - \kappa)\right) + \phi^A(\mu_Y - r - \sigma_Y\kappa - \pi - \delta) + K^{priv}(1 - \alpha(H_t)) \tag{48}$$

$$0 = \frac{\sigma_X^2}{2}\left(\phi^B\right)'' + \left(\phi^B\right)'\left(\mu_X + \sigma_X(\sigma_Y - \kappa)\right) + \phi^B(\mu_Y - r - \sigma_Y\kappa - \pi - \chi) + \psi\frac{1}{1-\upsilon}\frac{\pi+\chi}{\pi}\frac{\omega(X_t)}{g(X_t)}, \tag{49}$$

where we have used the simpler notation $\sigma_X, \mu_X, r, \kappa$, rather than $\sigma_X(X_t), \mu_X(X_t)$, etc. Finally, the functions β^i, $i \in \{A, B\}$, are given as $\beta^i(X_t) = \phi^i(X_t)/\zeta^i(X_t)$, where $\zeta^A(X_t)$ and $\zeta^B(X_t)$ solve the differential equations

$$-1 = \frac{\sigma_X^2}{2}\left(\zeta^A\right)'' + \left(\zeta^A\right)'\left(\mu_X - \sigma_X\frac{\gamma^A - 1}{\gamma^A}\kappa\right) - \zeta^A\left(\pi + r + \frac{\rho^A - r}{\gamma^A} + \frac{\gamma^A - 1}{\left(\gamma^A\right)^2}\frac{\kappa^2}{2}\right), \tag{50}$$

$$\begin{aligned}-1 &= \frac{\sigma_X^2}{2}\left(\zeta^B\right)'' + \left(\zeta^B\right)'\left(\mu_X + \sigma_X\frac{(1-\psi)\left(\gamma^B - 1\right)}{\gamma^B}\left(\sigma_Z + \frac{\omega'(X_t)}{\omega(X_t)}\sigma_X\right) - \sigma_X\frac{\gamma^B - 1}{\gamma^B}\kappa\right) \\ &\quad + \zeta^B\left[\mathcal{D}\left(w_t^{\frac{(1-\psi)\left(\gamma^B-1\right)}{\gamma^B}}\xi_t^{1-\frac{1}{\gamma^B}}\right) - \left(\pi + \frac{\rho^B}{\gamma^B}\right)\right].\end{aligned} \tag{51}$$

From the solution $\zeta^i(X_t)$, $\phi^i(X_t)$ to the differential equations above we obtain $\beta^i(X_t) = \phi^i(X_t)/\zeta^i(X_t)$, and then solve for r_t, κ_t, μ_X and σ_X as functions of X_t. As we illustrate in section 4.1, in some special cases it is even possible to obtain closed-form expressions for some of these quantities. Before doing this, we determine the value of the stock market.

3.4 Stock Price

Given ξ_t, it is possible to define the stock market value as follows

$$S_t \equiv E_t \int_t^\infty \left(\frac{\xi_u}{\xi_t}\right) D_u du. \tag{52}$$

To verify that the constructed allocation forms an equilibrium, it remains to verify conditions (17) and (18).[14] Adding up these two equations, and using Walras' law it suffices to verify that the aggregate financial wealth is equal to the stock market value

$$\upsilon \int_{-\infty}^t \pi e^{-\pi(t-s)} W_{t,s}^A ds + (1-\upsilon) \int_{-\infty}^t \pi e^{-\pi(t-s)} W_{t,s}^B ds = S_t \tag{53}$$

The next Lemma confirms that equation (53) holds.

Lemma 3 *Let S_t be defined as (52). Then equation (53) holds.*

It is also possible to give a simple expression for S_t in terms of the functions ζ^i, ϕ^i.

Lemma 4 *The stock market value is given as*

$$S_t = Y_t \left[\zeta^A(X_t) X_t + \frac{\zeta^B(X_t)}{\psi}(1-X_t) - \upsilon \phi^A(X_t) \frac{\pi}{\pi+\delta} - (1-\upsilon)\frac{\phi^B(X_t)}{\psi}\frac{\pi}{\pi+\chi}\right] \tag{54}$$

From (54) we obtain the price-dividend ratio as

$$p(X_t) = \frac{S_t}{D_t} = \frac{S_t}{Y_t K^{\text{pub}} \left[1 - \alpha\left(\mathcal{H}(X_t)\right)\right]}.$$

Finally, applying Ito's lemma to (54) together with the definitions of $p(X_t), D_t, Y_t$ gives the stock-market volatility and the stock market expected return as

$$\sigma_t = \sigma(X_t) = \sigma_Z + \left(\frac{p'}{p} - \frac{\alpha'\mathcal{H}'}{(1-\alpha)} + \frac{g'}{g}\right)\sigma_X \tag{55}$$

$$\mu_t = \mu(X_t) = r_t + \kappa_t \sigma_t. \tag{56}$$

[14] We assume throughout that $S_t < \infty$.

4 Qualitative Features of the Model

Before proceeding with an analysis of the quantitative implications of the model, it is instructive to start by examining some special cases that clarify the channels behind the model.

4.1 Overlapping Generations and the Risk-Free Rate Puzzle

We start our analysis with the special case $\gamma^A = \gamma^B = \gamma$, $\psi = 1$, and $\rho^A = \rho^B = \rho$, so that agents of type A and agents of type B have identical preferences, but different endowments. Then we obtain the following result.

Lemma 5 *An equilibrium is characterized by the following properties: (1) $H_t = 1$, and $\sigma_X = 0$, so that the evolution of X_t is deterministic. (2) κ_t is constant and given by $\kappa_t = \kappa = \gamma\sigma_Z$, (3) In steady state, the interest rate r is constant and is given by the solution to the equation*

$$\begin{aligned} 0 &= \frac{\pi + r + \frac{\rho - r}{\gamma} + \frac{\sigma_Z^2}{2}(\gamma - 1)}{r + \gamma\sigma_Z^2 + \pi + \delta - \mu_Z}(1 - K^{pub})(\pi + \delta)(1 - \alpha(1)) + \frac{\pi + r + \frac{\rho - r}{\gamma} + \frac{\sigma_Z^2}{2}(\gamma - 1)}{r + \gamma\sigma_Z^2 + \pi + \chi - \mu_Z}(\pi + \chi)\alpha(1) \\ &\quad + \frac{r - \rho}{\gamma} + \frac{\sigma_Z^2}{2}(\gamma + 1) - (\pi + \mu_Z). \end{aligned} \tag{57}$$

Figure 1 plots the solution to (57) and illustrates the effects of changing χ and δ on the equilibrium interest rate for various degrees of risk aversion. There are two conclusions that follow from Figure 1. First, the interest rate declines as either χ or δ increases. Second, this decline in the interest rate can be substantive. The graph reconfirms (in a stochastic environment) the observations originally made by Blanchard (1985) (in a deterministic setting): If agents face a downward sloping profile of earnings over the life cycle (say, because of retirement), interest rates are substantially lower in equilibrium, even for high levels of γ.

This is intuitive. In the standard infinite-horizon representative-agent framework, there is little need to save for the future, as the representative agent is entitled to the aggregate endowment both presently and in the future. In contrast, in an OLG framework an agent with income constituting a decreasing fraction of the aggregate endowment over her life cycle will need to save in order to sustain her standard of living. This effect increases savings and hence reduces the equilibrium interest rate.

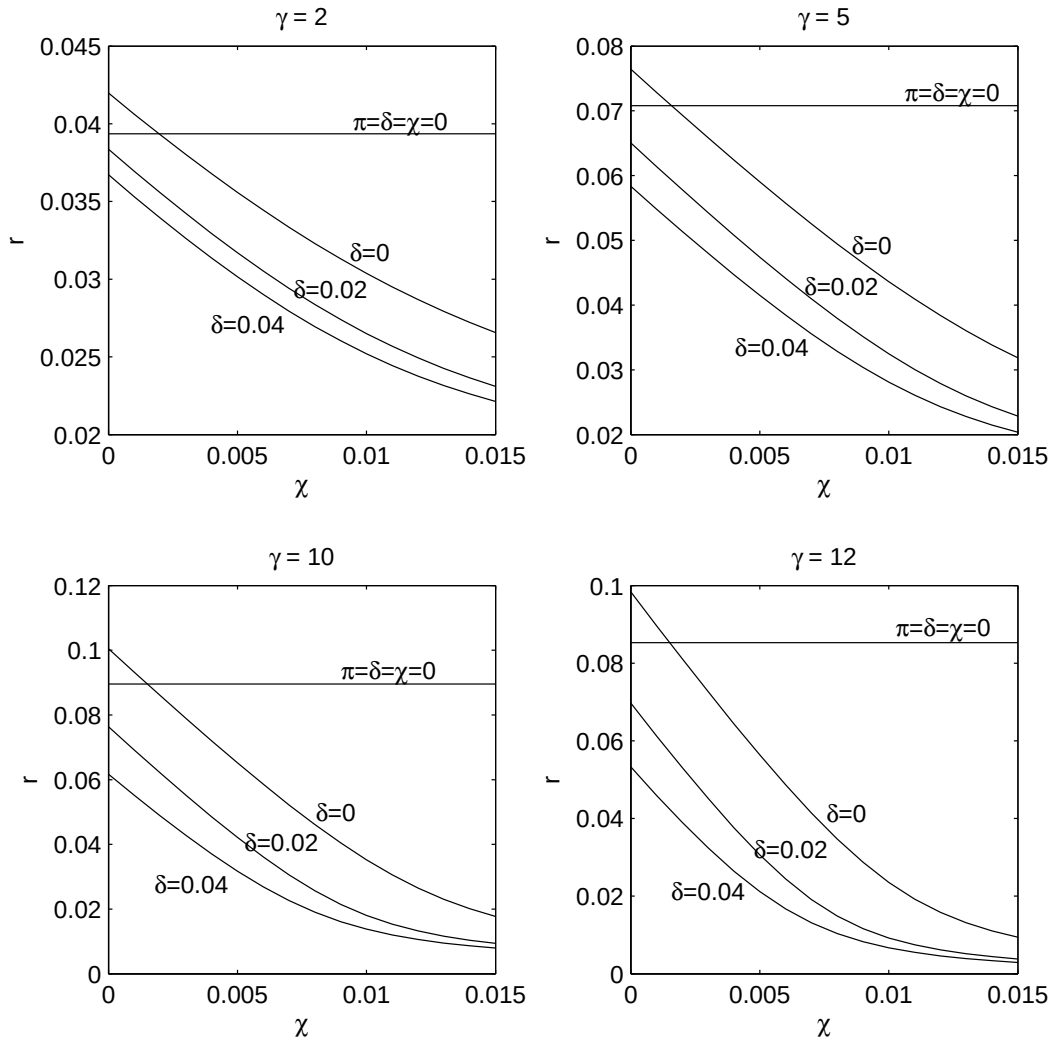

FIGURE 1: Equilibrium interest rates for different levels of χ, δ, and $\gamma = \gamma^A = \gamma^B$. The rest of the parameters are $\rho = 0.01$, $\pi = 0.01$, $\mu = 0.0172$, $\sigma = 0.041$, $\alpha(1) = 0.75$, and $K^{\text{pub}} = 0.4$. The horizontal line denoted by $\pi = \chi = \delta = 0$ corresponds to the infinite-horizon representative-agent model.

Even though the decline of income over the life cycle can help explain the low real rates that are observed in reality, the fact that agents have identical preferences implies that the price to dividend ratio is constant in the long run. Hence, the model cannot explain

why the stock market is more volatile than dividends, which in turn are more volatile than consumption. Next, we utilize the heterogeneity of preferences to introduce variation in discount rates and the price-to-dividend ratio.

4.2 Heterogenous Agents

4.2.1 Sharpe Ratio

In what follows we keep the same assumptions as in section 4.1, except that we allow for the possibility that $\gamma^A < \gamma^B$. This special case is particularly attractive for comparison purposes, because now X_t is a time-varying stochastic process with a non-degenerate stationary distribution. However, since $\psi = 1$, equation (26) continues to imply that $H_t = 1$. Since hours are constant, this implies that the aggregate output Y_t satisfies $dY_t/Y_t = dZ_t/Z_t$, so that the aggregate endowment follows a geometric Brownian motion, as is commonly assumed in the literature. Furthermore, since hours are not time varying, both $g(X_t) = f(\mathcal{H}(X_t))$ and $\omega(X_t) = f'(\mathcal{H}(X_t))$ are constant functions, and so is the fraction of output that accrues to labor, i.e., $\alpha(\mathcal{H}(X_t)) = \alpha(1)$.

We start our analysis by defining $\Gamma(X_t)$ as the weighted harmonic average of agents' risk aversions:

$$\Gamma(X_t) \equiv \frac{1}{\frac{X_t}{\gamma^A} + \frac{(1-X_t)}{\gamma^B}}. \tag{58}$$

Using this definition and the fact that hours are constant, equation (45) simplifies to

$$\kappa_t = \Gamma(X_t)\,\sigma_Z. \tag{59}$$

Since $\Gamma'(X_t) < 0$, if follows that κ_t is a declining function of X_t. Furthermore, equations (42) and (59) lead to the following expression for σ_X:

$$\frac{\sigma_X}{X_t} = \sigma_Z \left(\frac{\Gamma(X_t)}{\gamma^A} - 1 \right). \tag{60}$$

Since $X_t \in [0, 1]$, equation (58) implies that $\Gamma(X_t) > \gamma^A$, so that $\sigma_X \geq 0$. Hence, the state variable X_t increases in response to positive innovations to the exogenous productivity

process Z_t and hence to positive news about the aggregate endowment Y_t. Since κ_t is declining in X_t, this implies that the Sharpe ratio in the economy is countercyclical.

This property of the model is a first illustration of the consequences of aggregation: Less risk-averse agents (type-A agents) are more exposed to to aggregate productivity risks. As a result, their (total) wealth increases more than the (total) wealth of more risk-averse agents (type-B agents) in response to positive economic news, and so does their relative importance in the economy, which is captured by X_t. Furthermore, by equation (59), the Sharpe ratio is proportional to the (harmonic) weighted average of the risk aversions of the two agents, where the weights are given by X_t and $1 - X_t$. Accordingly, the Sharpe ratio declines in response to good news.

Interpreting $\Gamma(X_t)$ as the risk aversion of the "representative agent" shows the analogy to Campbell and Cochrane (1999). Even though in our framework each agent has constant relative risk aversion, equation (59) shows that assets are priced *as if* there exists a single representative agent with countercyclical risk aversion. Importantly, the interaction of heterogeneity with overlapping generations ensures that X_t is stationary, a property that may fail in economies with heterogeneous but infinitely lived agents.[15] Stationarity of X_t implies stationarity of valuation ratios, interest rates etc., making it possible to calibrate the model to the data.

4.2.2 Interest Rate

One of the most challenging facts for asset pricing models is that interest rates do not seem to vary as much as risk premia.[16] One of the main innovations of the Campbell and Cochrane (1999) model was to engineer a consumption-based stochastic discount factor where precau-

[15]The reason is intuitive. Given a positive risk premium, and the fact that less risk averse agents hold more stock implies that the expected growth rate of the less risk averse agents' wealth is higher than that of the more risk averse. Over sufficiently long horizons, this leads to the "extinction" of the more risk averse agents.

[16]For instance, the original habit-formation models (see e.g. Abel (1990), Constantinides (1990)) were successful at reproducing high equity premia, but at the cost of fairly volatile interest rates.

tionary savings motives exactly cancel variations in the intertemporal smoothing motive, so that interest rates remain constant.

In the present model, it is possible to maintain interest-rate volatility at low levels, but through a different channel. To gain intuition on this issue and simplify the formulae, we make the same parametric assumptions as in section 4.2.1. Substituting (59) into (46) and solving for r_t gives

$$r_t = \rho + \Gamma(X_t)\left[\mu_Z + \pi\left(1 - \upsilon\beta^A(X_t) - (1-\upsilon)\beta^B(X_t)\right)\right] - \frac{\sigma_Z^2}{2}\Gamma^3(X_t)\Delta(X_t), \tag{61}$$

where $\Delta(X_t)$ is defined as

$$\Delta(X_t) \equiv \left\{\frac{X_t}{\gamma^A}\left(\frac{\gamma^A+1}{\gamma^A}\right) + \frac{1-X_t}{\gamma^B}\left(\frac{\gamma^B+1}{\gamma^B}\right)\right\} = \frac{1}{\Gamma(X_t)} + \frac{X_t}{\gamma_A^2} + \frac{(1-X_t)}{\gamma_B^2}.$$

Notice that, for large values of γ^A and γ^B, $\Gamma(X_t)\Delta(X_t)$ is approximately equal to 1. [17]Hence (61) is approximated by

$$r_t \approx \rho + \Gamma(X_t)\mu_Z - \frac{\sigma_Z^2}{2}\Gamma^2(X_t) + \Gamma(X_t)\left[\pi\left(1 - \upsilon\beta^A(X_t) - (1-\upsilon)\beta^B(X_t)\right)\right]. \tag{62}$$

Equation (62) is similar to the expressions that are typically obtained in infinite horizon representative agent models. The first term on the right hand side of (62) is the subjective discount rate, the second term is related to the intertemporal smoothing motive, and the third term is related to the precautionary-savings motive of consumers. The major difference between the infinite horizon representative agent model and the present OLG model is the last term, and specifically the term inside the square brackets. Intuitively, this term adjusts for the fact that aggregate consumption growth and the consumption growth of the surviving agents (who matter for asset pricing) are not identical. On the one hand, agents that are alive at date t perish at the rate π, and this increases the expected growth rate of consumption to the survivors. On the other hand, new agents also arrive at the rate π, and claim a share $\upsilon\beta^A(X_t) + (1-\upsilon)\beta^B(X_t)$ of output Y_t as their consumption. Adding up the two effects gives the term inside square brackets in equation (62).

[17]Mathematically, $\lim_{\min(\gamma^A,\gamma^B)\to\infty}\Gamma(X_t)\Delta(X_t) = 1$.

Equation (62) is useful in showing how the model can produce low levels of interest-rate volatility. Collecting terms inside (62) gives

$$r_t \approx \rho + \Gamma(X_t)\left[\mu_Z - \frac{\sigma_Z^2}{2}\Gamma(X_t) + \pi\left(1 - \upsilon\beta^A(X_t) - (1-\upsilon)\beta^B(X_t)\right)\right]. \tag{63}$$

Equation (62) shows that in order to obtain interest rates that are low and non-volatile, there are two conditions that need to be satisfied. First, in order to obtain low interest rates, the term inside square brackets in equation (63) needs to be relatively small. Otherwise, the "representative" risk aversion $\Gamma(X_t)$ that premultiplies this term (and that typically is high, in order to account for the Sharpe ratio in the data) makes the interest rate implausibly large. Second, in order to obtain roughly constant interest rates, the term inside the square brackets in equation (63) needs to be increasing in X_t, since the term $\Gamma(X_t)$ that pre-multiplies the expression inside square brackets is declining in X_t.

The first property typically obtains with a declining labor income over the life cycle, as we discussed in section 4.1. The second property requires that $\upsilon\beta^A(X_t) + (1-\upsilon)\beta^B(X_t)$ is increasing in X_t, to counteract the fact that $\frac{\sigma_Z^2}{2}\Gamma(X_t)$ is declining in X_t .

Intuitively phrased, variations in X_t have two effects on the term inside square brackets in equation (62). On the one hand, they change the risk aversion of the representative agent, and hence the extent of precautionary savings. On the other hand, they affect the *survivors'* consumption growth rate, by determining the fraction of aggregate consumption that accrues to newly born agents of each type, namely $\beta^i(X_t)$. If an increase (decrease) in precautionary savings is counterbalanced by an increase (decrease) in *survivors'* consumption growth rates, then aggregate savings and interest rates will be roughly unaffected by changes in the representative agent's risk aversion. Hence, changes in X_t will affect mostly the Sharpe ratio and not the interest rate, as in the Campbell and Cochrane (1999) model.

In the next section we study circumstances under which $\upsilon\beta^A(X_t) + (1-\upsilon)\beta^B(X_t)$ is increasing in X_t.

4.3 Dividends and Labor

Since the two types of agents are endowed with different income streams, a first step towards examining $\upsilon\beta^A(X_t) + (1-\upsilon)\beta^B(X_t)$ is to investigate the properties of dividends and labor income in the model.

To obtain non-trivial variation in hours and hence in the labor and the dividend share of output we need to assume $\psi < 1$. Applying the implicit function theorem to (26) shows that

$$\frac{1}{H_t}\frac{dH_t}{dX_t} = \frac{\left(1-\frac{1}{\psi}\right)}{\alpha'(H_t)\left(1-H_t\right) - \frac{\alpha(H_t)}{H_t}}. \tag{64}$$

Since $\alpha'(H_t) \leq 0$, $\alpha(H_t) \geq 0$ and $H_t \leq 1$, the denominator on the right hand side of (64) is negative. When $\psi < 1$, then the numerator is also negative and hours are procyclical, i.e., an increasing function of X_t. The existence of stationary variation in hours implies that the model endogenously produces some cyclical variation in output alongside the variation caused by shocks to the (random walk) productivity process Z_t. Using the definition of g in equation (38), the fact that H_t is an increasing function of X_t also implies that output is increasing in X_t. Mathematically, $g'\left(X_t\right) > 0$.

Since we are interested in the asset-pricing implications of the model, we do not focus on these effects. Instead, we calibrate the model so as to ensure that hours supplied are roughly constant and as a result consumption is approximately a random walk. To achieve this, we choose ψ close to 1. The resulting small and procyclical variation in hours, in conjunction with our assumptions on the production function (equations [6] and [7]), results in a countercyclical labor share $\alpha(X_t)$ and hence a procyclical dividend share $1-\alpha(X_t)$ of output. More specifically, the definition of dividends for either publicly traded or privately held firms, together with (6), implies that the volatility of (log) dividends σ_D for either type of firm is given by

$$\sigma_D \equiv \sigma_Y - \frac{\alpha'\mathcal{H}'}{1-\alpha(\mathcal{H}(X_t))}\sigma_X. \tag{65}$$

Given the assumption $\psi < 1$, all three terms in the above expression are positive, since $\alpha' \leq 0$, $\sigma_X > 0$, and $\mathcal{H}' > 0$ by equation (64). This means that dividends are more volatile

than output (and hence consumption).[18] However, over longer horizons, (log) aggregate dividends and (log) output are cointegrated, since $(1 - \alpha(X_t))$ is stationary.

Another observation concerning (64) is that the extra volatility in dividends is driven by variation in X_t. (For instance if σ_X were 0, then $\sigma_D = \sigma_Z$). This dependence of the dividend share on variation in X_t translates into an increased sensitivity of the value of type A's endowment $\phi^i(X_t)$ to variation in X_t, since type A agents arrive in life with a dividend income stream. Since $\beta^i(X_t) = \phi^i(X_t)/\zeta^i(X_t)$, in calibrated versions of the model this increased sensitivity of ϕ^A to variation in X_t is sufficiently strong to ensure that $\upsilon\beta^A(X_t)+(1-\upsilon)\beta^B(X_t)$ is increasing in X_t. This is particularly the case when the entrepreneurial and less risk averse type-A agents save and accumulate more assets over their lifetimes, so that it is mostly the variation in $\beta^A(X_t)$ that drives the variation in $\upsilon\beta^A(X_t) + (1 - \upsilon)\beta^B(X_t)$.

To conclude our discussion of dividends and labor, we also note that the assumption of a countercyclical labor share together with a countercyclical Sharpe Ratio makes the model consistent with the three observations about labor-income growth reported in Lustig and Van Nieuwerburgh (2007). First, dividend growth and labor-income growth are negatively correlated in our framework.[19] Second, anticipated labor-income growth is positively correlated with "current" shocks to the productivity process.[20] Third, periods of high expected return coincide with periods of low anticipated income growth.[21]

[18]The property $\sigma_Y < \sigma_D$ follows from (40).

[19]Hence, when $\alpha(X_t)$ is above (below) its stationary mean, it can be expected to decline (increase). Therefore dividends can be expected to increase (decline) as a fraction of the aggregate endowment, while labor income can be expected to decline (increase).

[20]Shocks to the productivity shock Z_t increase X_t (by equation [60]) and hence make $\alpha(X_t)$ decline since $\alpha'(X_t) \leq 0$. Because $\alpha(X_t)$ can be expected to mean revert after such a shock, anticipated labor-income growth is positively correlated with "current" shocks to the productivity process.

[21]Periods of high expected returns occur when X_t is below its stationary mean. Therefore $\alpha(X_t)$ is above its stationary mean and can be expected to mean revert.

μ_Z	0.0172	γ^A	4	β_1	0.98
σ_Z	0.041	γ^B	12	β_2	0.60
π	0.01	ψ	0.95	β_3	1.2
χ	0.0054	ρ^A	0.001	β_4	0.34
δ	0.08	ρ^B	0.001	K^{pub}	0.4
υ	0.1				

TABLE 1: Parameters used in the calibration.

5 Quantitative Results

5.1 Parameter Choice and Calibration

To calibrate the model we need to choose eleven parameters along with a functional form for $\alpha(H_t) = \alpha(\mathcal{H}(X_t))$.

The parameters that we use for the calibration are given in Table 1. The parameters μ_Z and σ_Z are chosen so as to match the mean growth rate and the volatility of consumption growth respectively. As in Chan and Kogan (2002), we choose the volatility of instantaneous consumption to be higher than in yearly discrete-time data, in order to account for the effects of time aggregation.

The parameter π is chosen so that the median agent dies at age 69. As we have already discussed in section 2.1, χ can be interpreted as the arrival rate of a health shock that eliminates an agent's ability to work.[22] By letting $\chi = 0.0054$, a type B agent has on average the ability to work for $\frac{\pi}{\pi+\chi} = 65\%$ of her lifetime. Admittedly, the stylized assumptions of the model make it hard to calibrate π and χ. In real life neither π nor χ are constant. Nevertheless, given the tractability of aggregation that is allowed by these assumptions, we believe that our choices for π and χ are reasonable quantitatively.

The model assumes that a type-A agent arrives with an idea that is modeled as a unit of entrepreneurial capital stock. The parameter δ controls the economic depreciation of

[22] See in particular footnote 7.

such entrepreneurial capital - a parameter that is not immediately observable in the data. We use a common choice for depreciation by setting $\delta = 0.08$. This choice implies that entrepreneurial income is on average higher but also more ephemeral than labor income, so that type A agents save more than type B agents. Therefore, we can reproduce and study the general equilibrium consequences of the empirical finding[23] that entrepreneurial saving is higher than non-entrepreneurial saving. Moreover, this assumption is consistent with the fact that entrepreneurial businesses seem to have a short life-span in the data[24].

The parameter υ controls the fraction of the population that is comprised of the less risk averse / entrepreneurial agents. We set that number to 10%, which is close to the fraction of entrepreneurs in the population as reported in Gentry and Hubbard (2002).[25] K^{pub} controls the fraction of the capital stock that is owned by publicly traded corporations. To choose a value of K^{pub}, we use annual NIPA data since 1929. In NIPA data the ratio of "income from assets" to "proprietors' income" is roughly one to one. Interpreting proprietors' income as proceeds of entrepreneurial equity, this suggests a value of K^{pub} close to one half. However, to allow for the possibility that some entrepreneurial businesses are incorporated we choose $K^{\text{pub}} = 0.4$.

The parameters that pertain to agent preferences are given in the second column of Table 1. We set type-A agents' risk aversion to 4 and type-B agents' risk aversion to 12, with two motivations in mind. First, we want to match the average Sharpe ratio in the data and, second, we want to match the evidence in Malloy, Moskowitz, and Vissing-Jorgensen (2007), who argue that the "long-run" covariance between consumption and returns for wealthier, stock-holding households is up to 4 times larger than the equivalent covariance for the rest of the households. One can show that in our setup type-A agents have a covariance between long run consumption growth and returns that is $\gamma^B/\gamma^A = 3$ times higher than the equivalent quantity for type-B agents, well within the reasonable range of values supported by the data.

[23] See in particular Gentry and Hubbard (2002).

[24] See Gentry and Hubbard (2002).

[25] Gentry and Hubbard (2002) report numbers between 8.7% and 11.5% depending on the definition used to classify entrepreneurs.

For parsimony, we choose $\rho^A = \rho^B = 0.001$. The low value of the discount rate is motivated by the fact that in our OLG framework the "effective" discount rate is $\rho + \pi = 0.011$.

Finally, the parameter ψ that controls type-B agents' disutility of work is intentionally chosen very close to one. As we have discussed, equation (26) implies that when $\psi \approx 1$, hours do not vary considerably, and hence the predictable components of consumption growth become negligible. In particular, by combining values of $\psi \approx 1$ with a steeply declining $\alpha\left(H(X_t)\right)$ we can ensure that the volatility in X_t affects almost exclusively the dividend share of output, and not the predictable components of consumption. To have enough flexibility to obtain these properties, we parameterize $\alpha(H(X_t))$ as

$$\alpha(H(X_t)) = (\beta_1 - \beta_2)\,\mathcal{N}\left(\beta_3(\beta_4 - X)\right) + \beta_2, \tag{66}$$

where $\mathcal{N}$ is the cumulative normal distribution function and β_1, β_2, β_3, and β_4 are constants that we can choose to match certain properties of the data. Equation (66) implies that $\alpha \in (\beta_1, \beta_2)$ for any value of X_t, so that β_1 and β_2 control the range of α. The constants β_3 and β_4 control the steepness of the function and the point at which it achieves its maximum slope (in absolute value).

Our choices of β_1, β_2, β_3, and β_4 control the production function of the economy, and hence the dividend and labor shares of output. We choose these numbers so as to approximately match (i) the average dividend and labor share of national income and (ii) the year-to-year volatility of the dividend share. In computing this dividend share we include both proprietors' income and income from assets.[26]

	Data	Model
Mean annual consumption growth	1.7	1.7
Volatility of consumption growth	3.3	3.4
Mean riskless rate	2.9	2.5
Standard deviation of riskless rate	3.0	0.2
Mean equity premium (logarithmic returns)	3.9	2.6
Standard deviation of stock market returns	18.0	10.8
Sharpe ratio	0.22	0.28
Mean dividend share of output	24.4	22.1
Standard deviation of first differenced dividend share	0.9	0.9
Standard deviation of dividend growth rate	11.8	8.4

TABLE 2: Unconditional annual moments of the data. With the exception of the fourth, the eighth, and the ninth row, all data are from the long sample of Campbell and Cochrane (1999). The volatility of the interest rate is from Chan and Kogan (2002) and it refers to the volatility of the ex-post real rate. Hence, it overstates the volatility of the ex-ante riskless rate, because it doesn't account for inflation surprises. The eighth and ninth row are from NIPA data provided by the Bureau of Economic Analysis, spanning the years 1929-2005. We add proprietors' income and income from assets and divide by national income in computing the "dividend share" in the data. Simulation data are based on 10,000 simulated years of data for publicly traded companies, using a time increment of one month. Data are time-aggregated to yield yearly data.

5.2 Unconditional Moments

Table 2 compares the model's performance with some key moments in the data. As Mehra and Prescott (1985) and Chan and Kogan (2002), we calibrated the model to the century

[26]However, we excluded rental income of persons. The model has no role for land, and the asset pricing focus of the paper implies that one should match as closely as possible the fraction of income that is paid out to entrepreneurs and holders of corporate claims in the economy. Including rental income of land as part of the income that accrues to shareholders raises the "dividend share" from 24.4 percent to 29.7. As a robustness check, we also calibrated the model to this higher number, and obtained similar results.

long-sample data on consumption and returns that are reported in Campbell and Cochrane (1999).[27] The model's performance is not as good as in Campbell and Cochrane (1999), but it does explain a significant fraction of some asset-pricing facts. Most moments are within a reasonable distance from their empirical counterparts. The main moment that is underpredicted by the model is the volatility of equity.

There are three remarks about these results.

First, the model does not assume any financial leverage. We have made this assumption in order to investigate how much dividend volatility can be generated by the procyclicality of the dividend share. As Table 2 shows, the resulting dividend volatility is 8.4%, which is about 70% of the dividend volatility in the data. Hence, it should not be surprising that the model can only replicate 60% of the volatility of asset prices and 2/3 of the equity premium. Even though throughout the paper we compare the model's predictions to the observed stock prices in the data, it is worth noting that in the data equity is levered and the debt to equity ratio averages 0.56 in postwar data. Therefore, as Barro (2006) shows[28] the (implied) equity premium on unlevered equity is $1/(1+0.56) \approx 64\%$ of the (observed) equity premium on levered equity. A similar argument applies to the volatility of unlevered equity. Given the model's simplifying assumption that firms have no financial leverage, it seems reasonable to compare the imputed unlevered equity prices in the data to the equity prices implied by the model. Such a comparison shows that the model matches closely the (imputed) unlevered equity premium and volatility implied by the data. Having noted the distinction between levered and unlevered equity, for the rest of the paper we compare the

[27] Besides providing more accurate estimates of moments in the data, the century-long sample captures turbulent times for the US economy, such as the great depression. Moreover, as Siegel (1992) has shown with data since 1800, the post-world war II period is unusual by historical standards for the behavior of real interest rates and equity premia. However, the drawback of the long sample is that consumption data prior to world-war II are likely to be measured with some error. Nevertheless, even after accounting for such measurement error, there is broad agreement that macroeconomic data in the interwar period are substantially more volatile than post world-war II data, and there is debate only about the 30 years prior to world-war I. For a discussion, see Romer (1999) and Blake and Gordon (1989).

[28] See Barro (2006), Section III.

model-implied (unlevered) equity returns to the raw data on equity returns, without making any adjustments for leverage.

Second, in both the model and the data the volatility of stock prices is higher than the volatility of dividends, which in turn is substantially higher than the volatility of consumption. As we discuss in the next section, this implies that the variability of anticipated discount rate changes is larger than the variability of anticipated dividend growth, a fact with important implications for the predictability of excess returns.

Third, the stationary standard deviation of the interest rate is about 20 basis points in the model, i.e., the interest rate is practically constant. This implies that the variability in discount rates is almost exclusively driven by changes in excess returns, not interest rates.

The main conclusion is that the model explains a significant fraction of the unconditional asset-pricing moments commonly studied, despite the usage of standard expected-utility specifications and without relying on excessive interest-rate volatility.

5.3 Conditional Moments

Figure 2 gives a depiction of the instantaneous Sharpe ratio, risk-free rate, conditional volatility, and equity premium as functions of X_t. The range of values of X_t corresponds to ± 1 (stationary) standard deviations around its stationary mean. Figure 2 shows that the range of values for the conditional equity premium is substantially larger than the equivalent range for the riskless rate. Hence, it confirms that most of the variation in discount rates is related to variations of the equity premium, not the interest rate, consistent with the data.

Figure 3 addresses another feature of the model that is consistent with the data and presents a challenge for many models: the joint presence of a procyclical dividend share[29] and a procyclical price-to-dividend ratio. Figure 3 presents the P/D ratio as a function of X_t. Note that the P/D is increasing in X_t. This picture implies that there is time variation in anticipated changes to discount rates, and that this time variation is sufficiently strong to offset the time variation in anticipated dividend growth rates.

[29]Since α is decreasing in X_t, $1-\alpha$ is increasing in X_t

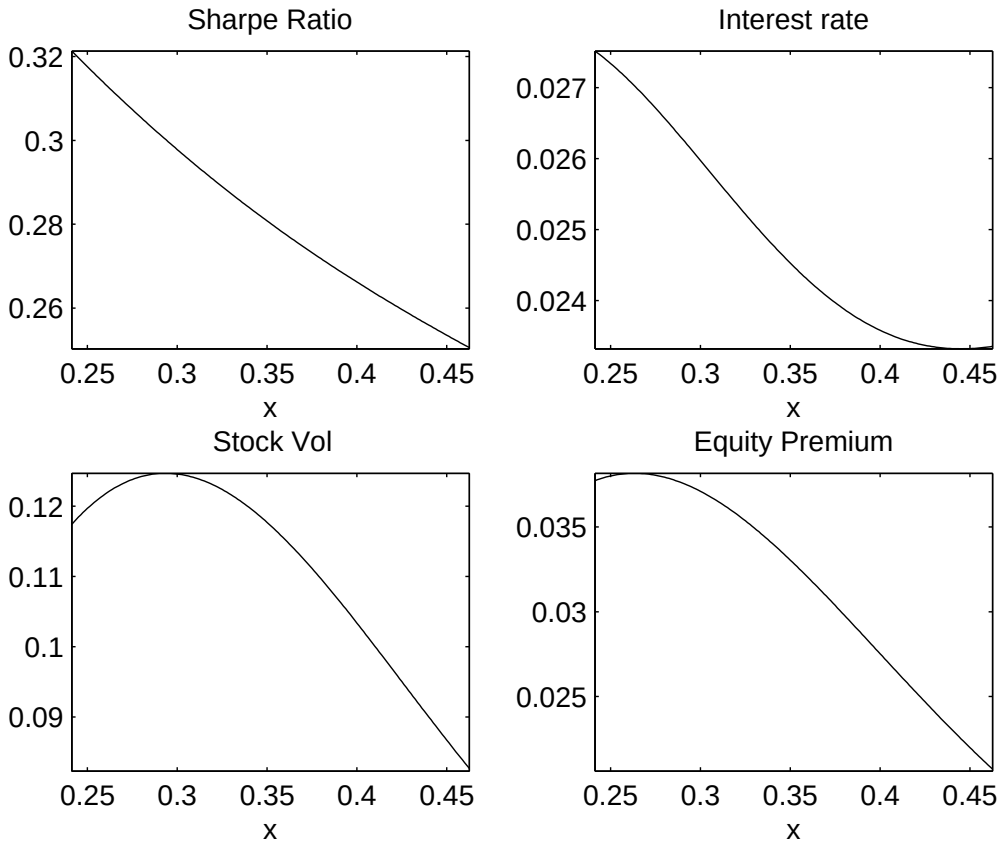

FIGURE 2: The four panels depict the Sharpe ratio, the interest rate, the stock volatility and the equity premium respectively as a function of the consumption share of type A agents (less risk averse agents), which is denoted as X. The range of X values corresponds approximately to ± 1 standard deviations around the stationary mean of X.

To see this point more clearly, consider figure 4. To simplify matters, assume that consumption is a random walk in logs, which is a close approximation to our model. The figure shows the effects of a 1 percent increase in current consumption. Since this effect is permanent, it shifts permanently the path of consumption at all future dates. Furthermore, since the dividend share is procyclical, dividends rise by $k > 1$ in the short run. However, because of co-integration, we also know that in the long run dividends will rise by 1 percent. These simple observations imply that in response to a positive shock to aggregate consumption, the

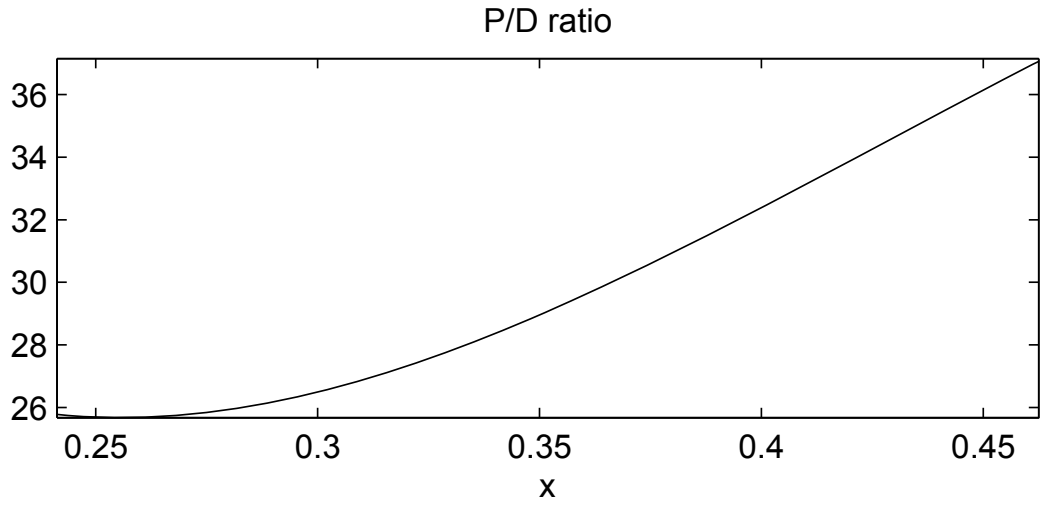

FIGURE 3: Price-to-dividend ratio as a function of the consumption share of type-A agents (less risk averse agents), which is denoted as X

anticipated growth rate of dividends will decline instead of increase, as is illustrated by the declining line in figure 4. If discount rates *were constant,* this would imply that the price-to-dividend ratio would decline in response to good news. Alternatively put, in response to a one percent increase in consumption, prices would rise by a percentage strictly smaller than k and, hence, dividends would be more volatile than asset prices.

In our setup, however, discount rates are not constant: Instead, they decline in response to positive shocks, as explained in section 4.2.1. In the calibrated version of the model, the decline in the anticipated dividend growth is smaller than the decline in discount rates, and thus the price-to-dividend ratio is procyclical, as in the data. This is consistent with the evidence in Lettau and Ludvigson (2005), who point out that the comovement of discount rates with the anticipated growth rate in dividends can help account for the observed inability of the price-to-dividend ratio to predict dividend growth.

Additionally, the model can help explain the finding that the price-dividend ratio typically predicts dividend growth with the wrong sign. (See e.g. Cochrane (2005), p. 392). Table 3 shows a direct implication of figure 4. Since both the P/D ratio and the dividend share are increasing in X_t, mean reversion in X_t implies that high P/D ratios anticipate a moderation

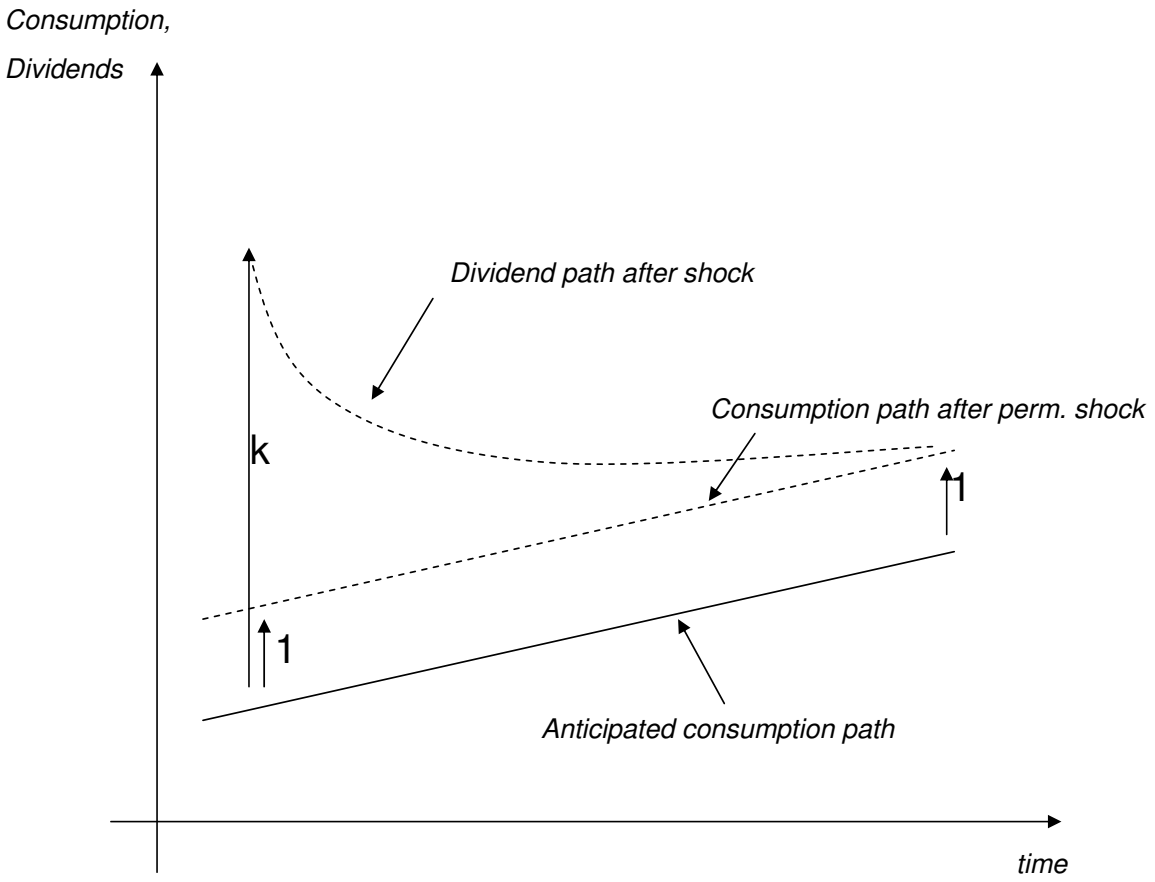

FIGURE 4: The implications of co-integration between dividends and consumption.

in dividend growth. Table 3 shows how this can help explain the puzzling finding that Campbell-Shiller type decompositions in the data assign a *negative* fraction of the variation in the P/D ratio to changes in dividend growth.

Table 4 gives yet a different perspective on these effects by showing the strong predictive ability of the P/D ratio for excess returns. The model overpredicts the absolute value of the coefficients in the predictive regressions for excess returns. This is partly driven by the fact that the model underpredicts the volatility of the (log) P/D ratio. This is to be expected, because of the offsetting effects of dividend growth on the time variation in discount rates. The R^2 of the regression, which is less affected by this issue, has the right order of magnitude

	Data	Model
Dividends	-10	-35
Returns	101	128

TABLE 3: Campbell-Shiller Variance Decompositions. The data are from the long sample of Campbell and Cochrane (1999). Simulation data are based on 10,000 simulated years of data for publicly traded companies. Time increments are monthly and are time-aggregated to yearly data.

Horizon (Years)	Data (Long Sample)		Model	
	Coefficient	R^2	Coefficient	R^2
1	-0.13	0.04	-0.20	0.02
2	-0.28	0.08	-0.38	0.05
3	-0.35	0.09	-0.58	0.07
5	-0.60	0.18	-0.96	0.11
7	-0.75	0.23	-1.37	0.16

TABLE 4: Long Horizon Regressions of excess returns on the log P/D ratio. To account for the well documented finite sample biases that are driven by the high autocorrelation of the P/D ratio, the simulated data are based on 1000 independent simulations of 100-year long samples, where the initial condition for X_0 for each of these simulation paths is drawn from the stationary distribution of X_t. For each of these 100-year long simulated samples, we run predictive regressions of the form $R_{t\to t+h} = \alpha + \beta \log(P_t/D_t)$, where h is the horizon for returns in years. We report the median values for the coefficient β and the R^2 of these regressions.

when compared with the data.

As an additional "goodness of fit" test, we also estimated the autocorrelation of the P/D ratio, and compared it to the data. To account for well known finite-sample biases, we adopted the same design setup as in Table 4, and estimated this autocorrelation in repeated 100-year long simulated samples. We obtained a median autocorrelation coefficient of 0.94.[30]

[30]The 95% coverage interval for this parameter was $[0.79, 1.01]$.

In the data, Campbell and Cochrane (1999) report an autocorrelation of 0.87 based on CRSP data and an autocorrelation of 0.78 based on the S&P sample. Therefore, even though the autocorrelation of the P/D ratio in the model is somewhat higher than in the data, it seems to be roughly consistent with the empirical magnitude of persistence.

6 Discussion and Extensions

6.1 The Dynamics of Cross-Sectional Inequality

The focus of the present model is on asset pricing. Nevertheless, it is interesting to check that the asset-pricing predictions of the model are not driven by counterfactual dynamics of cross-sectional consumption and wealth inequality.

As a first pass, the model's key mechanism, namely that the wealthier (type A) agents become comparatively richer when the stock market performs well and poorer agents become comparatively poorer when the stock market performs badly, is consistent with the data.[31] However, does the model need "too much" high-frequency variation in cross-sectional inequality to explain returns? This question is a common concern in all asset-pricing models that feature heterogeneity, since high-frequency changes in inequality are small.

Figure 5 depicts the stationary distribution of the share of consumption of type-A agents (X_t) in the top panel and the diffusion coefficient σ_X in the bottom panel. Intuitively, the bottom panel depicts how much one can expect the consumption share of type-A agents to change from year to year. The bottom panel of Figure 5 shows that the consumption share of type-A agents changes by about ± 1 percent from year to year. In simulations we find that this translates into a yearly change in the Gini coefficient of consumption inequality of

[31] Wolff (1992) provides some direct evidence to that effect. With data that go back to the twenties he shows that the wealth distribution becomes more uneven in response to positive excess stock market returns, even after controlling for changes in the income distribution. Furthermore, Malloy, Moskowitz, and Vissing-Jorgensen (2007) present evidence that the consumption share of shareholders is useful in predicting subsequent excess returns.

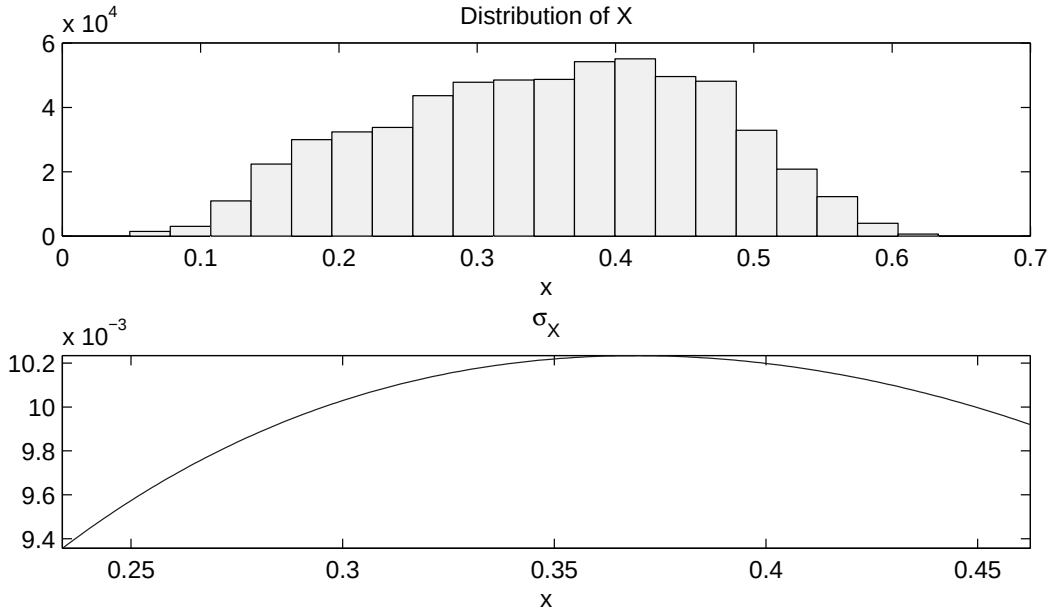

FIGURE 5: Stationary distribution (top figure) and instantaneous volatility (bottom figure) of X_t.

about 0.68 percent. This is consistent with the numbers given in Cutler and Katz (1992), based on repeated CES samples in the seventies and eighties.[32]

To understand how small year-to-year changes in inequality can have substantial effects on asset prices it is important to examine the persistence of X_t. Figure 5 depicts the stationary distribution of X_t. There are two observations from this figure. First, type-A agents account for about 37% of aggregate consumption, even though they are just 10% of the population.

[32]Cutler and Katz (1992) report the Gini coefficient for *consumption inequality* for the years 1960, 1972, 1980, 1984, and 1988. Computing the differences in Gini coefficients between those years, dividing by the square root of the time distance between the years (to account for heteroskedasticity in the observations) and then computing the standard deviation gives 0.70 percent. As a check that this number has the right order of magnitude, we also fitted an AR(1) to yearly post-war CPS Gini coefficient data on *income inequality*, since yearly consumption inequality data are not available for such a long period of time. We isolated the residuals and estimated their standard deviation to obtain 0.56 percent. Even though this number is for income rather than consumption inequality, it confirms that the year-to-year variability of inequality that is produced by the model is reasonable.

Second, the stationary distribution of X_t is substantially more "spread out" than the year-to-year increments to X_t. This illustrates that the state variable X_t is highly persistent. This is consistent with the data, where cross-sectional inequality behaves almost like a random walk.[33]

We conclude by noting that, despite the model's main focus on asset pricing, it makes two additional predictions about inequality that are consistent with the data: a) Inequality increases over the life cycle as documented by Deaton and Paxson (1994), and b) Type-A agents (entrepreneurs) have higher saving rates than type-B agents (non-entrepreneurs), as has been documented by Gentry and Hubbard (2002). Specifically, Deaton and Paxson (1994) impute a Gini coefficient for consumption inequality of 0.29 for 25-year olds and 0.38 for 55-year olds. In our model the equivalent Gini coefficients are about 0.20 for 25-year olds and about 0.24 for 55-year olds. What drives this increase in consumption inequality over the life cycle is that type-A agents both save more and bear more aggregate risk than type-B agents. The latter effect is clearly due to their lower risk aversion and it implies that type-A agents' wealth increases faster than that of type-B agents', due to the positive equity premium. The higher savings owe to the combination of an income profile that decays faster for type-A than for type-B agents ($\delta > \chi$) and an intertemporal elasticity of substitution[34] that is higher for type-A than for type-B agents $\left(1/\gamma^A > 1/\gamma^B\right)$.

Clearly, the model is too simple to facilitate a thorough study of consumption and income inequality. Such a theory would require the introduction of time-varying idiosyncratic shocks. However, as this section has shown, the model's prediction for inequality are qualitatively consistent with the data. We believe that the introduction of time-varying idiosyncratic

[33]The Gini coefficient of *income inequality* in the CPS data behaves almost like a random walk (autocorrelation coefficient 0.996). This is approximately equal to the autocorrelation of X_t in the model, which is about 0.99. Admittedly, this comparison between the model and the data is not exact, because the data refer to income inequality, whereas X_t is about consumption inequality. However, as Cutler and Katz (1992) argue, consumption and income inequality behave similarly over longer horizons.

[34]For expected utility preferences, the intertemporal elasticity of substitution is simply the inverse of risk aversion.

shocks would further strengthen the mechanisms of the paper, since entrepreneurs would have further incentives to save for precautionary reasons. However, such a model would be substantially less tractable.[35]

Even though a model with both time varying idiosyncratic and aggregate risk is unlikely to be tractable, it is straightforward to introduce individual-specific and time-invariant differences in earnings. The next subsection shows how to introduce such differences, and discusses how the uncertainty associated with such idiosyncratic effects induces less risk averse agents to become entrepreneurs and more risk averse agents to become workers.

6.2 The Link Between Low Risk Aversion and Entrepreneurship

Our baseline model simply assumes that agents with low risk aversion are entrepreneurs and agents with high risk aversion are workers, so that there is no occupational choice. Here we sketch how to extend the model in a simple way to allow for occupational choice without changing any of the asset pricing implications of the model.

Specifically, assume that at time s (i.e., the time of their birth) agents can choose whether to become entrepreneurs or workers.[36] Since the purpose of this subsection is illustrative, it will simplify matters to assume that this choice is irreversible (say because costs to switching professions are prohibitively large). Also, extend the model to allow for an individual-specific entrepreneurial skill Θ^i, so that an entrepreneur's human capital is given by $\Theta^i e^{-\delta(t-s)} K^{\text{priv}}_{s,s}$. The shock Θ^i is drawn from a log-normal distribution with mean 1 and variance Σ. Im-

[35] Angeletos and Calvet (2006) present such a model, which is analytically tractable, and show that idiosyncratic risk raises precautionary savings. However, they have to assume constant absolute risk aversion and no aggregate uncertainty, so that their model cannot be used for quantitatively studying the pricing of aggregate risk.

[36] If a type A agent chooses to become a worker, then she supplies all hours inelastically, since she has no utility for leisure. If a type B agent chooses to become an entrepreneur, then she can "purchase "hours on the market, since being an entrepreneur gives her no endowment of hours that can be used for leisure. An interpretation of "purchasing" hours is that an agent can hire outside help for household tasks, if she chooses to become an entrepreneur.

portantly, the realization of this random variable becomes known at time s^+, *after and only if* the agent made her choice to become an entrepreneur. Finally, suppose that becoming an entrepreneur is associated with a (multiplicative) utility benefit of $\left(U^E\right)^{(1-\gamma^i)}$, where $U^E > 1$. That is, the ratio of an agent's felicity function assuming that she chooses to become an entrepreneur to the same agent's felicity if she chooses to become a worker is given by $\left(U^E\right)^{(1-\gamma^i)}$ for $i \in A, B$. A motivation for this assumption is that agents (of both types) don't like to work for others, and enjoy the independence of entrepreneurship.

In such a modified setup it is possible to show that[37] there always exist values of Σ and $U^E > 1$, such that type-A agents always choose to become entrepreneurs and type-B agents always choose to become workers, given the equilibrium prices of section 3. The intuition is that the uncertainty introduced by Θ^i deters more risk averse agents from entrepreneurship, but not type-A agents.

Even though this simple extension makes the link between risk aversion and entrepreneurship endogenous, it does not affect any of the asset pricing implications of the model. The reason is simple and intuitive. Because of homogeneity, the consumption of entrepreneur i scales proportionately with Θ^i, which is not time varying. Since Θ^i has a mean of 1, the consumption share that accrues to entrepreneurs (i.e., type-A agents) *collectively* is not affected by Θ^i shocks. Therefore, the dynamics of X_t are unchanged and the equilibrium prices of Section 3 still clear markets.

7 Conclusion

In this paper we have presented a model that addresses a number of stylized facts about asset prices. The model combines three key ingredients: a) Agents are finitely lived, b) they can be heterogeneous in their preferences and endowments and c) consumption and dividends may differ.

These natural assumptions help explain simultaneously several asset-pricing phenomena:

[37] To keep the paper within a manageable size, we make the proof of this statement available upon request.

a) Risk-less rates are low, since life-cycle motivations enhance agents' incentive to save. b) The Sharpe ratio is volatile, since variations in the wealth and consumption distribution make our model resemble an economy populated by a representative agent with time-varying and countercyclical risk aversion. c) Since dividends are procyclical, more volatile than consumption, and discount rates vary countercyclically, stock market prices are volatile and the equity premium is reasonably high. d) Most of the variation in discount rates is due to changes in equity premia, not interest rates. e) The price-to-dividend ratio predicts excess returns. f) Even though dividends are predictable, the time variation in expected dividend growth is offset by changes in the stochastic discount factor, making the P/D ratio procyclical. g) Dividends are more volatile than consumption in the short run, but the two quantities are cointegrated over the long run. h) Consumption is practically a random walk.

These facts are consistent with the data. Moreover, calibrated versions of the model produce a satisfactory, albeit not perfect, quantitative fit.

Accordingly, we believe that the broad conclusion of the model is that overlapping generations along with preference and endowment heterogeneity can go a long way towards explaining prevailing asset-pricing puzzles. Observationally, our framework resembles a model of exogenous habit formation of the type proposed by Campbell and Cochrane (1999). However, the economic mechanisms of the models differ fundamentally. In our model, the stylized asset pricing facts emerge as a result of aggregating standard life cycle consumption problems.

Last but not least, an important advantage of the model is its analytic tractability. It provides us with a simple way of reproducing some key asset-pricing facts in a framework that can be used in various applications. For instance, the model could be expanded to investigate the effect of demographic shocks (such as a baby boom) on asset prices within a model that reproduces key asset-pricing facts. The conventional utilities that we use also facilitate policy experiments, such as the effects of a switch from pay-as-you-go to a fully funded system. We leave such extensions and applications for future research.

A Appendix

Proof of Lemma 1. Re-write Φ_s^A as $\Phi_s^A = \phi^A Y_s$, where

$$\phi^A = K^{\text{priv}} E_s \int_s^\infty e^{-(\pi+\delta)(t-s)} \left(\frac{Y_t}{Y_s}\right) (1-\alpha(H_t)) \left(\frac{\xi_t}{\xi_s}\right) dt. \tag{67}$$

Now note that (24) implies that $\frac{Y_t}{Y_s} = \frac{Z_t}{Z_s}\frac{f(H_t)}{f(H_s)} = \frac{f(H_t)}{f(H_s)} e^{\left(\mu_Z - \frac{\sigma_Z^2}{2}\right)(t-s)+\sigma_Z(B_t-B_s)}$. By (26) H_t is a function of X_t, while the exponential $e^{\left(\mu_Z - \frac{\sigma_Z^2}{2}\right)(t-s)+\sigma_Z(B_t-B_s)}$ does not depend on Y_t. Similarly, because r_t and κ_t are functions of X_t only, $\frac{\xi_t}{\xi_s}$ does not depend on Y_t. Finally $(1-\alpha(H_t))$ is independent of Y_t, since H_t is a function of X_t. Therefore, ϕ^A is independent of Y_t. Since ϕ^A is an expected integral of elements that depend on future values of X_t and X_t is Markovian (by assumption), it follows that ϕ^A is a function of X_s.

A similar argument shows that Φ_s^B can be written as $\phi^B Y_s$, where

$$\phi^B \equiv \psi \frac{\pi+\chi}{\pi} E_s \int_s^\infty e^{-(\pi+\chi)(t-s)} \frac{w_t}{Y_t} \left(\frac{\xi_t}{\xi_s}\right) dt. \tag{68}$$

Noting that $w_t = Z_t f'(H_t)$ and $Y_t = Z_t f(H_t)$ shows that w_t/Y_t is a function of H_t and hence of X_t only. Therefore, the same arguments as the ones we gave above show that ϕ^B is a function of X_t only. To show that $c_{s,s}^i = \beta^i(X_s) Y_s$ note that

$$E_s \int_s^\infty e^{-\pi(t-s)} c_{t,s}^i \left(\frac{\xi_t}{\xi_s}\right) dt = c_{s,s}^i \left[E_s \int_s^\infty e^{-\pi(t-s)} \frac{c_{t,s}^i}{c_{s,s}^i} \left(\frac{\xi_t}{\xi_s}\right) dt \right]. \tag{69}$$

Equations (32) and (31) imply that the term inside square brackets depends exclusively on X_t. Therefore equations (33) and (35) together with the fact that $\Phi_s^i = \phi^i(X_s) Y_s$ imply that $c_{s,s}^i = \beta^i(X_s) Y_s$. ■

Proof of Lemma 2. Fix a time s. For any $T < s$, equation (67) implies that

$$e^{-(\pi+\delta)s} \frac{\phi^A(X_s)}{K^{\text{priv}}} Y_s \xi_s + \int_T^s e^{-(\pi+\delta)t} Y_t (1-\alpha(H_t)) \xi_t \, dt = E_s \int_T^\infty e^{-(\pi+\delta)t} Y_t (1-\alpha(H_t)) \xi_t dt.$$

The right hand side of this expression is a martingale, since it is a conditional expectation. Letting s denote the time argument, applying Ito's Lemma to the left hand side of the above equation and setting the drift coefficient equal to zero implies (48). Applying a similar argument to (68) leads to (49). To obtain the functions β^i,define

$$\zeta^i \equiv E_s \int_s^\infty e^{-\pi(t-s)} \frac{c_{t,s}^i}{c_{s,s}^i} \frac{\xi_t}{\xi_s} dt. \tag{70}$$

Next use (69) along with equation (33), and $\Phi^i = \phi^i Y_s$ to obtain $\beta^i(X_t) = \frac{\phi^i(X_t)}{\zeta^i(X_t)}$.Furthermore, equations (32) and (31) imply

$$\begin{aligned}
\zeta^A &= E_s \int_s^\infty e^{-\left(\pi+\frac{\rho^A}{\gamma^A}\right)(t-s)} \left(\frac{\xi_t}{\xi_s}\right)^{1-\frac{1}{\gamma^A}} dt. \\
\zeta^B &= E_s \int_s^\infty e^{-\left(\pi+\frac{\rho^B}{\gamma^B}\right)(t-s)} \left(\frac{w_t}{w_s}\right)^{\frac{(1-\psi)\left(\gamma^B-1\right)}{\gamma^B}} \left(\frac{\xi_t}{\xi_s}\right)^{1-\frac{1}{\gamma^B}} dt.
\end{aligned}$$

By applying a similar argument to the one given for ϕ^A, ϕ^B we arrive at (50), (51). ■

Proof of Lemma 3. Using (13), applying Ito's lemma to compute $d\left(e^{-\pi s}\xi_s W_s^i\right)$, integrating and using the transversality condition leads to

$$W_{t,s}^i = E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \left(c_{u,s}^i - y_{u,s}^i\right) du \tag{71}$$

We next observe that generations that are born after t neither consume, nor supply hours, nor own any wealth at time t. This means that for any $i \in \{A, B\}, W_{t,s}^i = 0$, $c_{t,s}^i = 0$, $y_{t,s}^i = 0$ as long as $s > t$. Combining this observation with (71) and using the short-hand notation $\upsilon^A = \upsilon, \upsilon^B = 1-\upsilon$, we obtain

$$\begin{aligned}
&\sum_{i\in\{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \upsilon^i W_{t,s}^i ds = \sum_{i\in\{A,B\}} \int_{-\infty}^{+\infty} \pi e^{-\pi(t-s)} \upsilon^i W_{t,s}^i ds \\
&= \sum_{i\in\{A,B\}} \pi e^{-\pi(t-s)} \upsilon^i \int_{-\infty}^{+\infty} \left(E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \left(c_{u,s}^i - y_{u,s}^i\right) du \right) ds \\
&= E_t \int_t^\infty \frac{\xi_u}{\xi_t} \left(\sum_{i\in\{A,B\}} \int_{-\infty}^{+\infty} \pi e^{-\pi(u-s)} \upsilon^i \left(c_{u,s}^i - y_{u,s}^i\right) ds \right) du \\
&= E_t \int_t^\infty \frac{\xi_u}{\xi_t} \left(\sum_{i\in\{A,B\}} \int_{-\infty}^{u} \pi e^{-\pi(u-s)} \upsilon^i \left(c_{u,s}^i - y_{u,s}^i\right) ds \right) du = \\
&= E_t \int_t^\infty \frac{\xi_u}{\xi_t} D_u du,
\end{aligned}$$

where the last line follows from the equilibrium condition (16), (15), (10) and the definition of D_t.

■

Proof of Lemma 4. As we have shown in Lemma 3,

$$S_t = \sum_{i\in\{A,B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \upsilon^i W^i_{t,s} ds \tag{72}$$

$$= \sum_{i\in\{A,B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \upsilon^i \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c^i_{u,s} du \right] ds \tag{73}$$

$$- \sum_{i\in\{A,B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \upsilon^i \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y^i_{u,s} du \right] ds,$$

where we have used the short-hand notation $\upsilon^A = \upsilon, \upsilon^B = 1-\upsilon$.We can compute the first term in (73) as

$$\sum_{i\in\{A,B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \upsilon^i \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c^i_{u,s} du \right] ds$$

$$= \sum_{i\in\{A,B\}} \upsilon^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c^i_{t,s} \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \frac{c^i_{u,s}}{c^i_{t,s}} du \right] ds$$

Now note that equation (31) and (32) implies that $c^i_{u,s}/c^i_{t,s}$ for $i \in \{A,B\}$ is independent of s[38] i.e., $\frac{c^i_{u,s}}{c^i_{t,s}} = \frac{c^i_{u,t}}{c^i_{t,t}}$. Using this observation together with (70) leads to

$$\sum_{i\in\{A,B\}} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \upsilon^i \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c^i_{u,s} du \right] ds =$$

$$= \sum_{i\in\{A,B\}} \upsilon^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c^i_{t,s} \zeta^i(X_t)\, ds = Y_t \left[\zeta^A(X_t) X_t + \zeta^B(X_t)(1-X_t) \right]. \tag{74}$$

Similarly we can compute the second term in (73) separately for each agent. By using (34) and $\Phi^i(X_t) = \phi^i(X_t) Y_t$, we obtain for agent A

$$\upsilon \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y^A_{u,s} du \right] ds =$$

$$= K^{\text{priv}} \upsilon \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} e^{-\delta(u-s)} Y_u \left(1-\alpha(H_u)\right) du \right] ds$$

$$= K^{\text{priv}} \upsilon \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} e^{-\delta(u-t)} Y_u \left(1-\alpha(H_u)\right) du \right] ds$$

$$= Y_t \phi^A(X_t) \left[\upsilon \int_{-\infty}^{t} \pi e^{-(\pi+\delta)(t-s)} ds \right] = Y_t \left[\phi^A(X_t) \frac{\upsilon\pi}{\pi+\delta} \right]. \tag{75}$$

[38] To see this, fix a time of birth s, apply equation (31) at two different points in time, say u and t, and then derive $c^i_{u,s}/c^i_{t,s}$ which is independent of s.

Similarly, by using (21) we obtain for agent B

$$\begin{aligned}
&(1-\upsilon)\int_{-\infty}^{t}\pi e^{-\pi(t-s)}\left[E_t\int_t^{\infty}e^{-\pi(u-t)}\frac{\xi_u}{\xi_t}y^B_{u,s}du\right]ds\\
&= (1-\upsilon)\int_{-\infty}^{t}\pi e^{-\pi(t-s)}\left[E_t\int_t^{\infty}e^{-\pi(u-t)}\frac{\xi_u}{\xi_t}\left(\frac{\pi+\chi}{\pi}e^{-\chi(u-s)}w_u-\frac{(1-\psi)}{\psi}c^B_{u,s}\right)du\right]ds\\
&= (1-\upsilon)\int_{-\infty}^{t}\pi e^{-\pi(t-s)}\left[E_t\int_t^{\infty}e^{-\pi(u-t)}\frac{\xi_u}{\xi_t}\left(\frac{\pi+\chi}{\pi}e^{-\chi(u-s)}\right)w_udu\right]ds \qquad (76)\\
&\quad -Y_t\frac{(1-\psi)}{\psi}\zeta^B(X_t)(1-X_t).
\end{aligned}$$

Using (36), the first term in (76) can be further rewritten as

$$\begin{aligned}
&(1-\upsilon)\int_{-\infty}^{t}\pi e^{-\pi(t-s)}\left[E_t\int_t^{\infty}e^{-\pi(u-t)}\frac{\xi_u}{\xi_t}\left(\frac{\pi+\chi}{\pi}e^{-\chi(u-s)}\right)w_udu\right]ds\\
&= (1-\upsilon)\int_{-\infty}^{t}e^{-(\pi+\chi)(t-s)}\left[E_t\int_t^{\infty}e^{-(\pi+\chi)(u-t)}\frac{\xi_u}{\xi_t}(\pi+\chi)w_udu\right]ds\\
&= (1-\upsilon)Y_t\left[\frac{\phi^B(X_t)}{\psi}\pi\left(\int_{-\infty}^{t}e^{-(\pi+\chi)(t-s)}ds\right)\right]=Y_t\frac{\pi(1-\upsilon)}{\pi+\chi}\frac{\phi^B(X_t)}{\psi} \qquad (77)
\end{aligned}$$

Combining (77) with (76), (74), and (75), we arrive at (54). ■

Proof of Lemma 5. Solving for σ_X and κ_t from equations (42) and (45) gives $\sigma_X=0$ and $\kappa_t=\gamma\sigma_Z$. Therefore, X_t is deterministic. Moreover, equation (26) implies that $H_t=1$ for all X. By (24) it follows that $\sigma_Y=\sigma_z, \mu_Y=\mu_Z$. Furthermore, since hours do not vary with X_t, it also follows that $g', g'', \omega', \omega''$ are all zero. Using these observations inside equations (43) and (46) gives

$$\begin{aligned}
\mu_X+\mu_ZX_t &= X_t\left[\frac{r_t-\rho}{\gamma}+\frac{\kappa_t^2}{2}\frac{\gamma+1}{\gamma^2}-\pi\right]+\upsilon\pi\beta^A_t \qquad (78)\\
\mu_Z &= \sum_{i\in\{A,B\}}\upsilon^i\pi\beta^i_t+\frac{r_t-\rho}{\gamma}+\frac{\kappa_t^2}{2}\frac{\gamma+1}{\gamma^2}-\pi. \qquad (79)
\end{aligned}$$

Given knowledge of $\beta^i_t=\beta^i(X_t)$, the above two equations form a system in r_t, μ_X. We are interested in determining the steady state value of r. Since $H_t=1$, equation (6) implies that $\frac{\omega(\mathcal{H}(X_t))}{g(\mathcal{H}(X_t))}\equiv\frac{f'(1)}{f(1)}=\frac{\alpha(1)}{1}=\alpha(1)$. Also, in steady state $\mu_X=0$, and hence equations (48) and (49) imply

$$\begin{aligned}
\phi^A &= \frac{K^{\text{priv}}(1-\alpha(1))}{r+\sigma_Z\kappa+\pi+\delta-\mu_Z},\ \phi^B=\frac{\frac{1}{1-\upsilon}\frac{\pi+\chi}{\pi}\alpha(1)}{r+\sigma_Z\kappa+\pi+\chi-\mu_Z} \qquad (80)\\
\zeta^A &= \zeta^B=\frac{1}{\pi+r+\frac{\rho-r}{\gamma}+\frac{\gamma-1}{\gamma^2}\frac{\kappa^2}{2}} \qquad (81)
\end{aligned}$$

Substituting $\kappa=\gamma\sigma_Z$ and $K^{\text{priv}}=\frac{1}{\upsilon}\frac{\pi+\delta}{\pi}\left(1-K^{\text{pub}}\right)$ inside (80) and (81), using $\beta^i(X_t)=\phi^i(X_t)/\zeta^i(X_t)$, and then substituting into (79) gives (57). ■

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