Hindawi Publishing Corporation
The Scientific World Journal
Volume 2016, Article ID 1673537, 9 pages
http://dx.doi.org/10.1155/2016/1673537

Research Article

Z-Distance Based IF-THEN Rules

R. A. Aliev,[1,2] O. H. Huseynov,[3] and R. X. Zulfugarova[3]

[1] *Joint MBA Program, USA, Azerbaijan, Azerbaijan State Oil and Industry University, 20 Azadlig Avenue, 1010 Baku, Azerbaijan*
[2] *Department of Computer Engineering, Near East University, Lefkoşa, Northern Cyprus, Mersin 10, Turkey*
[3] *Department of Computer Engineering, Azerbaijan State Oil and Industry University, 20 Azadlig Avenue, 1010 Baku, Azerbaijan*

Correspondence should be addressed to R. A. Aliev; raliev@asoa.edu.az

Received 18 December 2015; Revised 4 February 2016; Accepted 8 February 2016

Academic Editor: Vilém Novák

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Decision making, reasoning, and analysis in real-world problems are complicated by imperfect information. Real-world imperfect information is mainly characterized by two features. In view of this, Professor Zadeh suggested the concept of a Z-number as an ordered pair $Z = (A, B)$ of fuzzy numbers A and B, the first of which is a linguistic value of a variable of interest, and the second one is a linguistic value of probability measure of the first one, playing a role of its reliability. The concept of distance is one of the important concepts for handling imperfect information in decision making and reasoning. In this paper, we, for the first time, apply the concept of distance of Z-numbers to the approximate reasoning with Z-number based IF-THEN rules. We provide an example on solving problem related to psychological issues naturally characterized by imperfect information, which shows applicability and validity of the suggested approach.

1. Introduction

Decision making, reasoning, and analysis in real-world problems are complicated by imperfect information. Real-world imperfect information is mainly characterized by two features. On the one hand, real-world information is often described on a basis of perception, experience, and knowledge of a human being. In turn, these operate with linguistic description carrying imprecision and vagueness, for which fuzzy sets based formalization can be used. On the other side, perception, experience, and knowledge of a human being are not sources of the truth. Therefore, the reliability is a degree of a partial confidence of a human being, which is naturally partial. This partial reliability is also naturally imprecise and can be formalized as a fuzzy value of probability measure. In order to ground the formal basis for dealing with real-world information, Zadeh suggested the concept of a Z-number [1] as an ordered pair $Z = (A, B)$ of continuous fuzzy numbers used to describe a value of a random variable X, where A is a fuzzy constraint on values of X and B is a fuzzy reliability of A and is considered as a value of probability measure of A. Nowadays a series of works devoted to Z-numbers and their application in decision making, control, and other fields [2–13] exists. A general and computationally effective approach to computation with discrete Z-numbers is suggested in [14–16]. The authors provide motivation of the use of discrete Z-numbers mainly based on the fact that NL-based information is of a discrete framework. The suggested arithmetic of discrete Z-numbers includes basic arithmetic operations and important algebraic operations.

The concept of distance is one of the important concepts for decision making and reasoning [17, 18]. In this paper, we for the first time apply the concept of distance of Z-numbers to the approximate reasoning with Z-number based IF-THEN rules. An approximate reasoning refers to a process of inferring imprecise conclusions from imprecise premises [17–38]. As one can see, this process often takes place in various fields of human activity including economics, decision analysis, system analysis, control, and everyday activity. The reason for this is that information relevant to real-world problems is, as a rule, imperfect. According to Zadeh, imperfect information is information which in one or more respects is imprecise, uncertain, incomplete, unreliable, vague, or partially true [39]. We can say that in a wide sense approximate reasoning is reasoning with imperfect information.

The paper is structured as follows. In Section 2, we present some prerequisite material including definitions of a discrete fuzzy number, a discrete Z-number, and probability measure of a discrete fuzzy number. In Section 3, we propose several distance measures for Z-numbers. In Section 4, we describe the statement of the problem and the suggested approach to reasoning with Z-rules on the basis of distance of Z-numbers. In Section 5, we illustrate an application of the suggested approach to a real-world problem which involves modeling of psychological aspects. Section 6 concludes.

2. Preliminaries

2.1. Main Definitions

Definition 1 (a discrete fuzzy number [40–43]). A fuzzy subset A of the real line $\mathcal{R}$ with membership function $\mu_A : \mathcal{R} \to [0,1]$ is a discrete fuzzy number if its support is finite; that is, there exist $x_1, \ldots, x_n \in \mathcal{R}$ with $x_1 < x_2 < \cdots < x_n$, such that $\mathrm{supp}(A) = \{x_1, \ldots, x_n\}$ and there exist natural numbers s, t with $1 \le s \le t \le n$ satisfying the following conditions:

(1) $\mu_A(x_i) = 1$ for any natural number i with $s \le i \le t$;

(2) $\mu_A(x_i) \le \mu_A(x_j)$ for natural numbers i, j with $1 \le i \le j \le s$;

(3) $\mu_A(x_i) \ge \mu_A(x_j)$ for natural numbers i, j with $t \le i \le j \le n$.

Definition 2 (a discrete random variable and a discrete probability distribution [44]). A random variable, X, is a variable whose possible values x are outcomes of a random phenomenon. A discrete random variable is a random variable which takes only a countable set of its values x.

Consider a discrete random variable X with outcomes space $\{x_1, \ldots, x_n\}$. A probability of an outcome $X = x_i$, denoted $P(X = x_i)$, is defined in terms of a probability distribution. A function p is called a discrete probability distribution or a probability mass function if

$$P(X = x_i) = p(x_i), \tag{1}$$

where $p(x_i) \in [0,1]$ and $\sum_{i=1}^{n} p(x_i) = 1$.

Definition 3 (arithmetic operations over discrete random variables [44, 45]). Let X_1 and X_2 be two independent discrete random variables with the corresponding outcome spaces $X_1 = \{x_{11}, \ldots, x_{1i}, \ldots, x_{1n_1}\}$ and $X_2 = \{x_{21}, \ldots x_{2i}, \ldots, x_{2n_2}\}$ and the corresponding discrete probability distributions p_1 and p_2. The probability distribution of $X_{12} = X_1 * X_2$, $* \in \{+, -, \cdot, /\}$, is the convolution $p_{12} = p_1 \circ p_2$ of p_1 and p_2 which is defined for any $x \in \{x_1 * x_2 \mid x_1 \in X_1,\ x_2 \in X_2\}$, $x_1 \in X_1$, $x_2 \in X_2$, as follows:

$$p_{12}(x) = \sum_{x = x_1 * x_2} p_1(x_1)\, p_2(x_2). \tag{2}$$

Definition 4 (probability measure of a discrete fuzzy number [46]). Let X be discrete random variable with probability distribution p. Let A be a discrete fuzzy number describing a possibilistic restriction on values of X. A probability measure of A denoting $P(A)$ is defined as

$$\begin{aligned} P(A) &= \sum_{i=1}^{n} \mu_A(x_i)\, p(x_i) \\ &= \mu_A(x_1)\, p(x_1) + \mu_A(x_2)\, p(x_2) + \cdots \\ &\quad + \mu_A(x_n)\, p(x_n). \end{aligned} \tag{3}$$

Definition 5 (a scalar multiplication of a discrete fuzzy number [16]). A scalar multiplication of a discrete fuzzy number A by a real number $\lambda \in \mathcal{R}$ is the discrete fuzzy number $A_1 = \lambda A$, whose α-cut is defined as

$$\begin{aligned} &A_1^{\alpha} \\ &= \{x \in \lambda \cdot \mathrm{supp}(A) \mid \min(\lambda A^{\alpha}) \le x \le \max(\lambda A^{\alpha})\}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \lambda \cdot \mathrm{supp}(A) &= \{\lambda x \mid x \in \mathrm{supp}(A)\}, \\ \min(\lambda A^{\alpha}) &= \min\{\lambda x \mid x \in A^{\alpha}\}, \\ \max(\lambda A^{\alpha}) &= \max\{\lambda x \mid x \in A^{\alpha}\}, \end{aligned} \tag{5}$$

and the membership function is defined as

$$\mu_{\lambda A}(x) = \sup\{\alpha \in [0,1] \mid x \in (\lambda A^{\alpha})\}. \tag{6}$$

Definition 6 (addition of discrete fuzzy numbers [40–43]). For discrete fuzzy numbers A_1, A_2, their addition $A_{12} = A_1 + A_2$ is the discrete fuzzy number whose α-cut is defined as

$$\begin{aligned} A_{12}^{\alpha} = \{&x \in \{\mathrm{supp}(A_1) + \mathrm{supp}(A_2)\} \mid \min\{A_1^{\alpha} + A_2^{\alpha}\} \\ &\le x \le \max\{A_1^{\alpha} + A_2^{\alpha}\}\}, \end{aligned} \tag{7}$$

where $\mathrm{supp}(A_1) + \mathrm{supp}(A_2) = \{x_1 + x_2 \mid x_j \in \mathrm{supp}(A_j),\ j = 1,2\}$, $\min\{A_1^{\alpha} + A_2^{\alpha}\} = \min\{x_1 + x_2 \mid x_j \in A_j^{\alpha},\ j = 1,2\}$, $\max\{A_1^{\alpha} + A_2^{\alpha}\} = \max\{x_1 + x_2 \mid x_j \in A_j^{\alpha},\ j = 1,2\}$, and the membership function is defined as

$$\mu_{A_1 + A_2}(x) = \sup\{\alpha \in [0,1] \mid x \in \{A_1^{\alpha} + A_2^{\alpha}\}\}. \tag{8}$$

Definition 7 (a discrete Z-number [15, 16]). A discrete Z-number is an ordered pair $Z = (A, B)$ of discrete fuzzy numbers A and B. A plays a role of a fuzzy constraint on values that a random variable X may take. B is a discrete fuzzy number with a membership function $\mu_B : \{b_1, \ldots, b_n\} \to [0,1]$, $\{b_1, \ldots, b_n\} \subset [0,1]$, playing a role of a fuzzy constraint on the probability measure of A, $P(A) = \sum_{i=1}^{n} \mu_A(x_i) p(x_i)$, $P(A) \in \mathrm{supp}(B)$.

3. Distance between Two Z-Numbers

Denote by $\mathcal{F}$ the space of discrete fuzzy sets of $\mathcal{R}$. Denote by $\mathcal{F}_{[a,b]}$ the space of discrete fuzzy sets of $[a,b] \subset \mathcal{R}$.

Definition 8 (the supremum metric on $\mathscr{D}$ [47]). The supremum metric d on $\mathscr{F}$ is defined as

$$d(A_1, A_2) = \sup\{d_H(A_1^\alpha, A_2^\alpha) \mid 0 < \alpha \le 1\}, \quad A_1, A_2 \in \mathscr{F}, \tag{9}$$

where d_H is the Hausdorff distance.

$(\mathscr{F}, d)$ is a complete metric space [47, 48].

Definition 9 (fuzzy Hausdorff distance [16]). The fuzzy Hausdorff distance d_{fH} between $A_1, A_2 \in \mathscr{F}$ is defined as

$$d_{fH}(A_1, A_2) = \bigcup_{\alpha\in[0,1]} \alpha d_{fH}^\alpha(A_1, A_2), \tag{10}$$

where

$$d_{fH}^\alpha(A_1, A_2) = \left\{\sup_{\alpha\le\bar{\alpha}\le 1} d_H\left(A_1^{\bar{\alpha}}, A_2^{\bar{\alpha}}\right)\right\}, \tag{11}$$

where $\bar{\alpha}$ is the value which is within α-cut and 1-cut. $(\mathscr{F}, d_{fH})$ is a complete metric space.

Denote by $\mathscr{Z}$ the space of discrete Z-numbers:

$$\mathscr{Z} = \{Z = (A, B) \mid A \in \mathscr{F},\ B \in \mathscr{F}_{[0,1]}\}. \tag{12}$$

Definition 10 (supremum metrics on $\mathscr{Z}$ [16]). The supremum metrics on $\mathscr{Z}$ are defined as

$$D(Z_1, Z_2) = d(A_1, A_2) + d(B_1, B_2); \tag{13}$$

$(\mathscr{Z}, D)$ is a complete metric space. This follows from the fact that $(\mathscr{F}, d)$ is a complete metric space.

$D(Z_1, Z_2)$ has the following properties:

$$\begin{gathered} D(Z_1 + Z, Z_2 + Z) = D(Z_1, Z_2), \\ D(Z_2, Z_1) = D(Z_1, Z_2), \\ D(\lambda Z_1, \lambda Z_2) = |\lambda| D(Z_1, Z_2), \quad \lambda \in \mathscr{R}, \\ D(Z_1, Z_2) \le D(Z_1, Z) + D(Z, Z_2). \end{gathered} \tag{14}$$

Definition 11 (fuzzy Hausdorff distance between Z-numbers [16]). The fuzzy Hausdorff distance d_{fHZ} between Z-numbers $Z_1 = (A_1, B_1), Z_2 = (A_2, B_2) \in \mathscr{Z}$ is defined as

$$d_{fHZ}(Z_1, Z_2) = d_{fH}(A_1, A_2) + d_{fH}(B_1, B_2). \tag{15}$$

Definition 12 (Z-valued Euclidean distance between discrete Z-numbers [16]). Given two discrete Z-numbers $Z_1 = (A_1, B_1), Z_2 = (A_2, B_2) \in \mathscr{Z}$, Z-valued Euclidean distance $d_E(Z_1, Z_2)$ between Z_1 and Z_2 is defined as

$$d_E(Z_1, Z_2) = \sqrt{(Z_1 - Z_2)^2}. \tag{16}$$

4. Z-Valued IF-THEN Rules Based Reasoning

A problem of interpolation of Z-rules termed as Z-interpolation was addressed by Zadeh as a challenging problem [33]. This problem is the generalization of interpolation of fuzzy rules [49]. The problem of Z-interpolation is given below.

Given the following Z-rules,

if X_1 is $Z_{X_1,1} = (A_{X_1,1}, B_{X_1,1})$ and so on and X_m is $Z_{X_m,1} = (A_{X_m,1}, B_{X_m,1})$, then Y is $Z_Y = (A_{Y,1}, B_{Y,1})$,

if X_1 is $Z_{X_1,2} = (A_{X_1,2}, B_{X_1,2})$ and so on and X_m is $Z_{X_m,2} = (A_{X_m,2}, B_{X_m,2})$, then Y is $Z_Y = (A_{Y,2}, B_{Y,2})$,

if X_1 is $Z_{X_1,n} = (A_{X_1,n}, B_{X_1,n})$ and so on and X_m is $Z_{X_m,n} = (A_{X_m,n}, B_{X_m,n})$ then Y is $Z_Y = (A_{Y,n}, B_{Y,n})$,

and a current observation

X_1 is $Z'_{X_1} = (A'_{X_1}, B'_{X_1})$ and so on and X_m is $Z'_{X_m} = (A'_{X_m}, B'_{X_m})$,

find the Z-value of Y. Here m is the number of Z-valued input variables and n is the number of rules.

The idea underlying the suggested interpolation approach is that the ratio of distances between the resulting output and the consequent parts is equal to one between the current input and the antecedent parts [49]. This implies for Z-rules that the resulting output Z'_Y is computed as

$$Z'_Y = \sum_{j=1}^{n} w_j Z_{Y,j} = \sum_{j=1}^{n} w_j\left(A_{Y,j}, B_{Y,j}\right), \tag{17}$$

where $Z_{Y,j}$ is the Z-number valued consequent of the jth rule, $w_j = (1/\rho_j)/(\sum_{j=1}^{n} 1/\rho_j)$, $j = 1, \ldots, n$ are coefficients of linear interpolation, and n is the number of Z-rules. $\rho_j = \sum_{i=1}^{m} D(Z'_{X_i}, Z_{X_i,j})$, where D is the distance between current ith Z-number valued input and the ith Z-number valued antecedent of the jth rule. Thus, ρ_j computes the distance between a current input vector and the vector of the antecedents of jth rule.

In this paper, we will consider discrete Z-numbers. The operations of addition and scalar multiplication of discrete Z-numbers are described below.

Addition of Discrete Z-Numbers. Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be discrete Z-numbers describing imperfect information about values of variables X_1 and X_2. Consider the problem of computation of addition $Z_{12} = Z_1 + Z_2$. The first stage is the computation addition of discrete fuzzy numbers $A_{12} = A_1 + A_2$ on the basis of Definition 6.

The second stage involves stage-by-stage construction of B_{12} which is related to propagation of probabilistic restrictions. We realize that, in Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$, the "true" probability distributions p_1 and p_2 are not exactly known. In contrast, the information available is represented by the fuzzy restrictions:

$$\begin{gathered} \sum_{k=1}^{n_1} \mu_{A_1}(x_{1k})\, p_1(x_{1k}) \text{ is } B_1, \\ \sum_{k=1}^{n_2} \mu_{A_2}(x_{2k})\, p_2(x_{2k}) \text{ is } B_2, \end{gathered} \tag{18}$$

which are represented in terms of the membership functions as

$$\mu_{B_1}\left(\sum_{k=1}^{n_1}\mu_{A_1}(x_{1k})\,p_1(x_{1k})\right),\qquad \mu_{B_2}\left(\sum_{k=1}^{n_2}\mu_{A_2}(x_{2k})\,p_2(x_{2k})\right). \tag{19}$$

Thus, one has the fuzzy sets of probability distributions of p_1 and p_2 with the membership functions defined as

$$\mu_{p_1}(p_1)=\mu_{B_1}\left(\sum_{k=1}^{n_1}\mu_{A_1}(x_{1k})\,p_1(x_{1k})\right),\qquad \mu_{p_2}(p_2)=\mu_{B_2}\left(\sum_{k=1}^{n_2}\mu_{A_2}(x_{2k})\,p_2(x_{2k})\right). \tag{20}$$

Therefore, we should construct these fuzzy sets. B_j, $j=1,2$, is a discrete fuzzy number which plays the role of a soft constraint on a value of a probability measure of A_j. Therefore, basic values $b_{jl}\in\operatorname{supp}(B_j)$, $j=1,2$, $l=1,\ldots,m$, of a discrete fuzzy number B_j, $j=1,2$, are values of a probability measure of A_j, $b_{jl}=P(A_j)$. Thus, given b_{jl}, we have to find such probability distribution p_{jl} which satisfies

$$b_{jl}=\mu_{A_j}(x_{j1})\,p_{jl}(x_{j1})+\mu_{A_j}(x_{j2})\,p_{jl}(x_{j2})+\cdots+\mu_{A_j}(x_{jn_j})\,p_{jl}(x_{jn_j}). \tag{21}$$

At the same time, we know that p_{jl} has to satisfy

$$\sum_{k=1}^{n_j}p_{jl}(x_{jk})=1,\quad p_{jl}(x_{jk})\geq 0. \tag{22}$$

Thus, the following goal programming problem should be solved to find p_j:

$$\mu_{A_j}(x_{j1})\,p_{jl}(x_{j1})+\mu_{A_j}(x_{j2})\,p_{jl}(x_{j2})+\cdots+\mu_{A_j}(x_{jn_j})\,p_{jl}(x_{jn_j})\longrightarrow b_{jl}, \tag{23}$$

subject to

$$p_{jl}(x_{j1})+p_{jl}(x_{j2})+\cdots+p_{jl}(x_{jn_j})=1,\qquad p_{jl}(x_{j1}),p_{jl}(x_{j2}),\ldots,p_{jl}(x_{jn_j})\geq 0. \tag{24}$$

For each $l=1,\ldots,m$ and each $k=1,\ldots,n_j$ denote $c_k=\mu_{A_j}(x_{jk})$ and $v_k^l=p_{jl}(x_{jk})$, $k=1,\ldots,n_j$. As c_k and b_{jl} are known and v_k^l are unknown, we see that problem (23)-(24) is nothing but the following goal linear programming problem:

$$c_1v_1^l+c_2v_2^l+\cdots+c_nv_n^l\longrightarrow b_{jl}, \tag{23$'$}$$

subject to

$$v_1^l+v_2^l+\cdots+v_n^l=1,\qquad v_1^l,v_2^l,\ldots,v_n^l\geq 0. \tag{24$'$}$$

Having obtained the solution v_k^l, $k=1,\ldots,n_j$, of problems (23′)-(24′) for each $l=1,\ldots,m$, recall that $v_k^l=p_{jl}(x_{jk})$, $k=1,\ldots,n_j$. As a result, $p_{jl}(x_{jk})$, $k=1,\ldots,n_j$, is found, and, therefore, distribution p_{jl} is obtained. Thus, to construct $\mu_{p_{jl}}$, we need to solve m simple problems (23′)-(24′). Let us mention that in general, problems (23′)-(24′) do not have a unique solution. In order to guarantee existence of a unique solution, the compatibility conditions can be included:

$$\sum_{k=1}^{n_j}x_{jk}p_{jl}(x_{jk})=\frac{\sum_{k=1}^{n_j}x_{jk}\mu_{A_j}(x_{jk})}{\sum_{k=1}^{n_j}\mu_{A_j}(x_{jk})}. \tag{25}$$

This condition implies that the centroid of A_j is to coincide with that of p_{jl}.

Probability distributions $p_{jl}(x_{jk})$, $k=1,\ldots,n_j$, naturally induce probabilistic uncertainty over the result $X=X_1+X_2$. This implies, given any possible pair p_{1l}, p_{2l} of the extracted distributions, the convolution $p_{12s}=p_{1l}\circ p_{2l}$, $s=1,\ldots,m^2$, is to be computed as follows:

$$p_{12}(x)=\sum_{x_1+x_2=x}p_{1l}(x_1)\,p_{2l}(x_2),\qquad \forall x\in X_{12};\ x_1\in X_1,\ x_2\in X_2. \tag{26}$$

Given p_{12s}, the value of probability measure of A_{12} can be computed:

$$P(A_{12})=\sum_{k=1}^{n}\mu_{A_{12}}(x_{12k})\,p_{12}(x_{12k}). \tag{27}$$

However, the "true" p_{12s} is not exactly known as the "true" p_{1l} and p_{2l} are described by fuzzy restrictions. In other words, the fuzzy sets of probability distributions p_{1l} and p_{2l} induce the fuzzy set of convolutions p_{12s}, $s=1,\ldots,m^2$, with the membership function defined as

$$\mu_{p_{12}}(p_{12})=\max_{p_1,p_2}\left[\mu_{p_1}(p_1)\wedge\mu_{p_2}(p_2)\right], \tag{28}$$

subject to

$$p_{12}=p_1\circ p_2,\qquad \mu_{p_j}(p_j)=\mu_{B_j}\left(\sum_{k=1}^{n_j}\mu_{A_j}(x_{jk})\,p_{jl}(x_{jk})\right), \tag{29}$$

where $\wedge$ is min operation.

As a result, fuzziness of information on p_{12s} described by $\mu_{p_{12}}$ induces fuzziness of the value of probability measure $P(A_{12})$ as a discrete fuzzy number B_{12}. The membership function is defined as

$$\mu_{B_{12}}(b_{12s})=\sup\left(\mu_{p_{12s}}(p_{12s})\right), \tag{30}$$

subject to

$$b_{12s} = \sum_k p_{12s}(x_k)\,\mu_{A_{12}}(x_k). \tag{31}$$

As a result, $Z_{12} = Z_1 + Z_2$ is obtained as $Z_{12} = (A_{12}, B_{12})$.

Scalar Multiplication of Discrete Z-Numbers. Let us consider a scalar multiplication of a discrete Z-number $Z_X = (A_X, B_X)$: $Z_Y = \lambda \cdot Z_X$, $\lambda \in \mathscr{R}$. The resulting $Z_Y = (A_Y, B_Y)$ is found as follows. $A_Y = \lambda A_X$ is determined based on Definition 5.

In order to construct B_Y, at first probability distributions $p_{X,l}$, $l = 1, \ldots, m$, should be extracted by solving a linear programming problem analogous to (23′)-(24′). Next, we realize that $p_{X,l}$, $l = 1, \ldots, m$, induce probability distributions $p_{Y,l}$, $l = 1, \ldots, m$, related to Z_Y as follows:

$$p_Y = p_Y(y_1) \setminus y_1 + p_Y(y_2) \setminus y_2 + \cdots + p_Y(y_n) \setminus y_n, \tag{32}$$

such that

$$\begin{aligned} y_k &= \lambda x_k, \\ p_Y(y_k) &= p_X(x_k). \end{aligned} \tag{33}$$

The fuzzy set of probability distributions p_X with membership function $\mu_{p_X}(p_{X,l}) = \mu_{\widetilde{B}_X}(\sum_{k=1}^{n} \mu_{\widetilde{A}_X}(x_k) p_{X,l}(x_k))$ induces the fuzzy set of probability distributions $p_{Y,l}$ with the membership function defined as

$$\mu_{p_Y}(p_{Y,l}) = \mu_{p_X}(p_{X,l}), \tag{34}$$

taking into account (32)-(33).

Next, we compute probability measure of A_Y, given p_Y. Given a fuzzy restriction on p_Y described by μ_{p_Y}, we construct a fuzzy number B_Y with the membership function μ_{B_Y}:

$$\mu_{B_Y}(b_{Y,l}) = \sup\left(\mu_{p_Y}(p_{Y,l})\right), \tag{35}$$

subject to

$$b_{Y,l} = \sum_k p_{Y,l}(x_k)\,\mu_{A_Y}(x_k). \tag{36}$$

As a result, $Z_Y = \lambda \cdot Z_X$ is obtained as $Z_Y = (A_Y, B_Y)$.

Let us now consider the special case of the considered problem of Z-rules interpolation, suggested in [50, 51].

Given the Z-rules

$$\begin{gathered} \text{If } X \text{ is } A_{X,1} \text{ then } Y \text{ is } \left(A_{Y,1}, B\right) \\ \text{If } X \text{ is } A_{X,2} \text{ then } Y \text{ is } \left(A_{Y,2}, B\right) \\ \vdots \\ \text{If } X \text{ is } A_{X,n} \text{ then } Y \text{ is } \left(A_{Y,n}, B\right) \end{gathered} \tag{37}$$

and a current observation

$$X \text{ is } \left(A_X, B_X\right), \tag{38}$$

find the Z-value of Y.

For this case, as the reliabilities of the Z-number based consequents of the considered rules are equal, $B_{Y,k} = B$, according to formula (17) the Z-number valued output of the Z-rules, $Z'_Y = (A'_Y, B'_Y)$, is computed as

$$\begin{aligned} Z'_Y &= \sum_{j=1}^{n} w_j Z_{Y,j} = \sum_{j=1}^{n} w_j \left(A_{Y,j}, B_{Y,j}\right) \\ &= \sum_{j=1}^{n} w_j \left(A_{Y,j}, B\right), \end{aligned} \tag{39}$$

where $w_j = (1/\rho_j)/(\sum_{k=1}^{n} 1/\rho_k)$ and $\rho_j = \sum_{i=1}^{m} D(Z'_{X_i}, Z_{X_i,j}) = \sum_{i=1}^{m} D((A'_{X_i}, 1), (A_{X_i,j}, 1)) = \sum_{i=1}^{m} d(A'_{X_i}, A_{X_i,j})$ as both inputs and the antecedents of the considered Z-rules are of a special Z-number; that is, they are represented by discrete fuzzy numbers with the reliability equal to 1.

5. An Application

Let us consider modeling of a fragment of a relationship between the student motivation, attention, anxiety, and educational achievement [52]. The information on the considered characteristics is naturally imprecise and partially reliable. Indeed, one deals mainly with intangible, nonmeasurable mental indicators. For this reason, the use of Z-rules, as rules with Z-number valued inputs and outputs based on linguistic terms from a predefined codebook, is adequate way for modeling of this relationship. This rules will help to evaluate a student with given Z-number based evaluations of the characteristics. Consider the following Z-rules:

> *The 1st rule*: If motivation is (M, U), attention is (H, U), and anxiety is (L, U), then achievement is (E, U).
>
> *The 2nd rule*: If motivation is (M, U), attention is (M, U), and anxiety is (M, U), then achievement is (G, U).

Here, the pairs $(\cdot,\cdot)$ are Z-numbers where uppercase letters denote the following linguistic terms: H, High; L, Low; M, Medium; G, Good; E, Excellence; U, Usually. The codebooks containing linguistic terms of values of antecedents and consequents are given in Figures 1, 2, 3, and 4. The codebook for the degrees of reliability of values of antecedents and consequents is shown in Figure 5.

The considered Z-numbers are given below.

The 1st rule inputs:

$$\begin{aligned} Z_{A_M} &= \frac{0}{2.6} + \frac{0.5}{3.3} + \frac{1}{4} + \frac{0.5}{4.7} + \frac{0}{5.4}, \\ Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}; \\ Z_{A_H} &= \frac{0}{57.5} + \frac{0.5}{68.75} + \frac{1}{80} + \frac{1}{90}, \end{aligned}$$

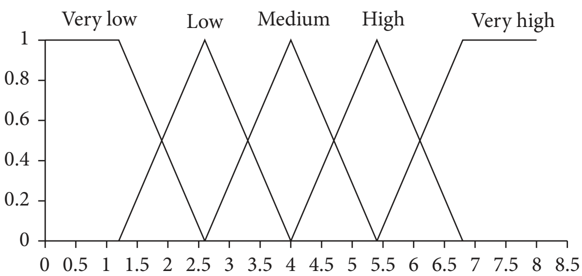


FIGURE 1: Linguistic terms for a value of motivation.

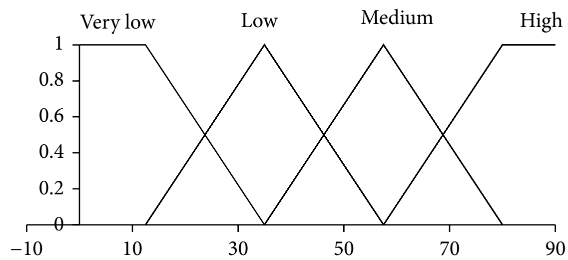


FIGURE 2: Linguistic terms for a value of attention.

$$
\begin{aligned}
Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9};\\
Z_{A_L} &= \frac{0}{1.19} + \frac{0.5}{1.6} + \frac{1}{2} + \frac{0.5}{2.4} + \frac{0}{2.8},\\
Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}.
\end{aligned}
\tag{40}
$$

The 1st rule output:

$$
\begin{aligned}
Z_{A_{VH}} &= \frac{0}{80} + \frac{0.5}{85} + \frac{1}{90} + \frac{0.5}{95} + \frac{0}{100},\\
Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}.
\end{aligned}
\tag{41}
$$

The 2nd rule inputs:

$$
\begin{aligned}
Z_{A_M} &= \frac{0}{2.6} + \frac{0.5}{3.3} + \frac{1}{4} + \frac{0.5}{4.7} + \frac{0}{5.4},\\
Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9};\\
Z_{A_M} &= \frac{0}{35} + \frac{0.5}{46.25} + \frac{1}{57.5} + \frac{0.5}{68.75} + \frac{0}{80},\\
Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9};\\
Z_{A_M} &= \frac{0}{2} + \frac{0.5}{2.4} + \frac{1}{2.8} + \frac{0.5}{3.2} + \frac{0}{3.6},\\
Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}.
\end{aligned}
\tag{42}
$$

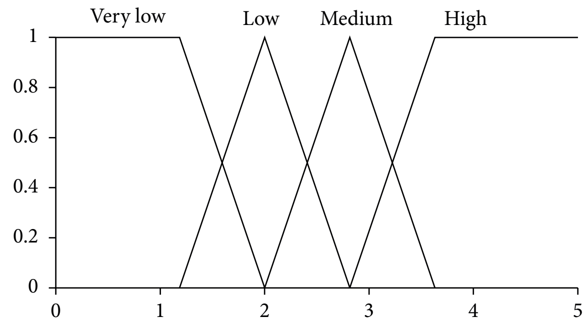


FIGURE 3: Linguistic terms for a value of anxiety.

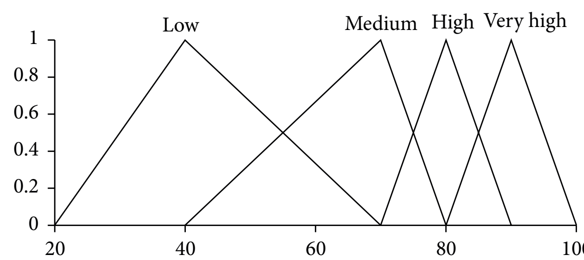


FIGURE 4: Linguistic terms for a value of achievement.

The 2nd rule output:

$$
\begin{aligned}
Z_{A_H} &= \frac{0}{70} + \frac{0.5}{75} + \frac{1}{80} + \frac{0.5}{85} + \frac{0}{90},\\
Z_{B_U} &= \frac{0}{0.7} + \frac{0.5}{0.75} + \frac{1}{0.8} + \frac{0.5}{0.85} + \frac{0}{0.9}.
\end{aligned}
\tag{43}
$$

Consider a problem of reasoning within the given Z-rules by using the suggested Z-interpolation approach. Let the current input information for motivation, attention, and anxiety be described by the following Z-numbers $Z_1 = (Z_{A_1}, Z_{B_1})$, $Z_2 = (Z_{A_2}, Z_{B_2})$, and $Z_3 = (Z_{A_3}, Z_{B_3})$, respectively:

$$
\begin{aligned}
Z_{A_1} &= \frac{0}{2.5} + \frac{0.5}{3} + \frac{1}{3.5} + \frac{0.5}{4} + \frac{0}{4.5},\\
Z_{B_1} &= \frac{0}{0.6} + \frac{0.5}{0.65} + \frac{1}{0.7} + \frac{0.5}{0.75} + \frac{0}{0.8};\\
Z_{A_2} &= \frac{0}{25} + \frac{0.5}{35} + \frac{1}{45} + \frac{0.5}{55} + \frac{0}{65},\\
Z_{B_2} &= \frac{0}{0.6} + \frac{0.5}{0.65} + \frac{1}{0.7} + \frac{0.5}{0.75} + \frac{0}{0.8};\\
Z_{A_3} &= \frac{0}{1.3} + \frac{0.5}{2.3} + \frac{1}{3.3} + \frac{0.5}{3.65} + \frac{0}{4},\\
Z_{B_3} &= \frac{0}{0.6} + \frac{0.5}{0.65} + \frac{1}{0.7} + \frac{0.5}{0.75} + \frac{0}{0.8}.
\end{aligned}
\tag{44}
$$

Z-interpolation approach based reasoning consists of two main stages.

(1) For each rule compute *dist* as distance ρ_j between the current input Z-information $Z_1 = (Z_{A_1}, Z_{B_1})$, $Z_2 = (Z_{A_2}, Z_{B_2})$, and $Z_3 = (Z_{A_3}, Z_{B_3})$ and Z-antecedents of Z-rules base $Z_{j1} = (A_{j1}, B_{j1})$, $Z_{j2} = (A_{j2}, B_{j2})$, and $Z_{j3} =$

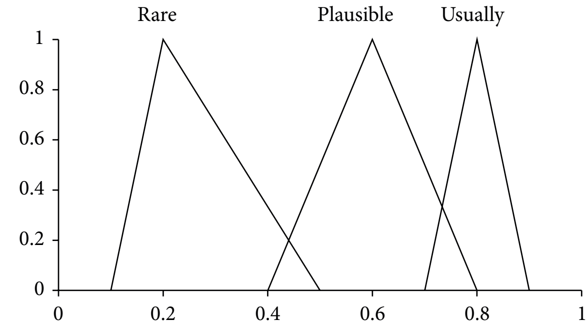


Figure 5: Linguistic terms for reliability of antecedents and consequents.

(A_{j3}, B_{j3}), $j = 1, 2$. For simplicity, we will use the supremum metric $D(Z_i, Z_{ji})$ (13):

$$\rho_j = \sum_{i=1}^{3} D\left(Z_i, Z_{ji}\right). \tag{45}$$

Consider computation of ρ_j for the 1st and 2nd rules. Thus, we need to determine $\rho_j = \sum_{j=1}^{3} D(Z_j, Z_{1j})$, where values $D(Z_1, Z_{11})$, $D(Z_2, Z_{12})$, and $D(Z_3, Z_{13})$ are computed on the basis of (13). We have obtained the results:

$$\begin{aligned} D\left(Z_1, Z_{11}\right) &= d_H\left(A_1, A_{11}\right) + d_H\left(B_1, B_{11}\right) \\ &= 0.9 + 0.1 = 1, \\ D\left(Z_2, Z_{12}\right) &= 40.1, \\ D\left(Z_3, Z_{13}\right) &= 1.4. \end{aligned} \tag{46}$$

Thus, the distance for the 1st rule is

$$\rho_1 = 42.5. \tag{47}$$

Analogously, we computed the distance for the 2nd rule as

$$\begin{aligned} D\left(Z_1, Z_{2,1}\right) &= 1, \\ D\left(Z_2, Z_{2,2}\right) &= 15.1, \\ D\left(Z_3, Z_{2,3}\right) &= 0.8, \\ \rho_2 &= 16.9. \end{aligned} \tag{48}$$

(2) Computation of the aggregated output Z_Y for Z-rules base by using linear Z-interpolation:

$$\begin{gathered} Z_Y = w_1 Z_{Y,1} + w_2 Z_{Y,2}, \\ w_1 = \frac{1/\rho_1}{1/\rho_1 + 1/\rho_2}, \quad w_2 = \frac{1/\rho_2}{1/\rho_1 + 1/\rho_2}. \end{gathered} \tag{49}$$

The obtained interpolation coefficients are $w_1 = 0.28$ and $w_2 = 0.72$. The aggregated output Z_Y is defined as

$$Z_Y = 0.28 Z_{Y_1} + 0.72 Z_{Y_2} = \left(A_Y, B_Y\right). \tag{50}$$

We have obtained the following result:

$$\begin{aligned} Z_{A_Y} &= \frac{0}{72.8} + \frac{0.5}{78.2} + \frac{1}{82.6} + \frac{0.5}{84} + \frac{0}{89}, \\ Z_{B_Y} &= \frac{0}{0.68} + \frac{0.5}{0.73} + \frac{1}{0.78} + \frac{0.5}{0.81} + \frac{0}{0.84}. \end{aligned} \tag{51}$$

In accordance with the codebooks shown in Figures 4 and 5, we have achievement is "*High*" with the reliability being "*Usually*." This linguistic approximation is made by using similarity measure between the obtained output and fuzzy sets in the codebooks.

6. Conclusion

A concept of a Z-number suggested by Zadeh is a key to computation with imprecise and partial reliable information. In this paper, we propose applying distance of Z-numbers to approximate reasoning within IF-THEN rules with Z-numbers-based antecedents and consequents.

A real-world application of the suggested research has been provided to illustrate its validity and potential applicability.

Competing Interests

The authors declare that they have no competing interests.

References

[1] L. A. Zadeh, "A note on Z-numbers," *Information Sciences*, vol. 181, no. 14, pp. 2923–2932, 2011.

[2] R. A. Aliev and L. M. Zeinalova, "Decision making under Z-information," in *Human-Centric Decision-Making Models for Social Sciences*, P. Guo and W. Pedrycz, Eds., Studies in Computational Intelligence, pp. 233–252, Springer, 2014.

[3] R. Banerjee and S. K. Pal, "The Z-number enigma: a study through an experiment," in *Soft Computing State of the Art Theory and Novel Applications*, R. R. Yager, A. M. Abbasov, M. Z. Reformat, and S. N. Shahbazova, Eds., vol. 291 of *Studies in Fuzziness and Soft Computing*, pp. 71–88, Springer, Berlin, Germany, 2013.

[4] B. Kang, D. Wei, Y. Li, and Y. Deng, "A method of converting Z-number to classical fuzzy number," *Journal of Information & Computational Science*, vol. 9, no. 3, pp. 703–709, 2012.

[5] B. Kang, D. Wei, Y. Li, and Y. Deng, "Decision making using Z-numbers under uncertain environment," *Journal of Computational Information Systems*, vol. 8, no. 7, pp. 2807–2814, 2012.

[6] E. S. Khorasani, P. Patel, S. Rahimi, and D. Houle, "An inference engine toolkit for computing with words," *Journal of Ambient Intelligence and Humanized Computing*, vol. 4, no. 4, pp. 451–470, 2013.

[7] J. Lorkowski, R. A. Aliev, and V. Kreinovich, "Towards decision making under interval, set-valued, fuzzy, and Z-number uncertainty: a fair price approach," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 2244–2253, Beijing, China, July 2014.

[8] S. K. Pal, S. Dutta, R. Banerjee, and S. S. Sarma, "An insight into the Z-number approach to CWW," *Fundamenta Informaticae*, vol. 124, no. 1-2, pp. 197–229, 2013.

[9] M. D. Springer, *The Algebra of Random Variables*, John Wiley & Sons, New York, NY, USA, 1979.

[10] S. Tadayon and B. Tadayon, "Approximate Z-number evaluation based on categorical sets of probability distributions," in *Recent Developments and New Directions in Soft Computing*, vol. 317 of *Studies in Fuzziness and Soft Computing*, pp. 117–134, Springer, Berlin, Germany, 2014.

[11] R. R. Yager, "On a view of Zadeh's Z-numbers," in *Advances in Computational Intelligence: 14th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2012, Catania, Italy, July 9–13, 2012, Proceedings, Part III*, vol. 299 of *Communications in Computer and Information Science*, pp. 90–101, Springer, Berlin, Germany, 2012.

[12] L. T. Kóczy and K. Hirota, "Approximate reasoning by linear rule interpolation and general approximation," *International Journal of Approximate Reasoning*, vol. 9, no. 3, pp. 197–225, 1993.

[13] L. A. Zadeh, "Methods and systems for applications with Z-numbers," United States Patent no. US 8,311,973 B1, November 2012.

[14] R. A. Aliev, A. V. Alizadeh, O. H. Huseynov, and K. I. Jabbarova, "Z-number-based linear programming," *International Journal of Intelligent Systems*, vol. 30, no. 5, pp. 563–589, 2015.

[15] R. A. Aliev, A. V. Alizadeh, and O. H. Huseynov, "The arithmetic of discrete Z-numbers," *Information Sciences*, vol. 290, pp. 134–155, 2015.

[16] R. A. Aliev, O. H. Huseynov, R. R. Aliyev, and A. A. Alizadeh, *The Arithmetic of Z-Numbers. Theory and Applications*, World Scientific, Singapore, 2015.

[17] L. T. Kóczy, "Approximate reasoning by linear rule interpolation and general approximation," *International Journal of Approximate Reasoning*, vol. 9, no. 3, pp. 197–225, 1993.

[18] L. A. Zadeh, "Interpolative reasoning in fuzzy logic and neural network theory," in *Proceedings of the 1st IEEE International Conference on Fuzzy Systems*, San Diego, Calif, USA, 1992.

[19] S. K. Pal and D. P. Mandal, "Fuzzy logic and approximate reasoning an overview," *The Journal of the Institution of Electronics and Telecommunication Engineers, India*, vol. 37, no. 5-6, pp. 548–560, 1991.

[20] D. Dubois and H. Prade, "Fuzzy sets in approximate reasoning, Part 1: inference with possibility distributions," *Fuzzy Sets and Systems*, vol. 100, no. 1, pp. 73–132, 1999.

[21] R. R. Yager, "On the theory of approximate reasoning," *Controle Automacao*, vol. 4, pp. 116–125, 1994.

[22] L. A. Zadeh, "Fuzzy logic and approximate reasoning," *Synthese*, vol. 30, no. 3-4, pp. 407–428, 1975.

[23] R. R. Yager, "Deductive approximate reasoning systems," *IEEE Transactions on Knowledge and Data Engineering*, vol. 3, no. 4, pp. 399–414, 1991.

[24] R. R. Yager, "Fuzzy sets and approximate reasoning in decision and control," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 415–428, IEEE, March 1992.

[25] K. K. Dompere, *Fuzziness and Approximate Reasoning. Epistemics on Uncertainty, Expectation and Risk in Rational Behavior*, vol. 237 of *Studies in Fuzziness and Soft Computing*, Springer, Berlin, Germany, 2009.

[26] R. Aliev and A. Tserkovny, "Systemic approach to fuzzy logic formalization for approximate reasoning," *Information Sciences*, vol. 181, no. 6, pp. 1045–1059, 2011.

[27] E. H. Mamdani, "Application of fuzzy logic to approximate reasoning using linguistic synthesis," *IEEE Transactions on Computers*, vol. 26, no. 12, pp. 1182–1191, 1977.

[28] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.

[29] L. A. Zadeh, "Outline of a new approach to the analysis of complex system and decision processes," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 3, pp. 28–44, 1973.

[30] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1975, Part II, vol. 8, no. 4, pp. 301–357, Part III, vol. 9, no. 1, pp. 43–80.

[31] L. A. Zadeh, "Toward a generalized theory of uncertainty (GTU)—an outline," *Information Sciences*, vol. 172, no. 1-2, pp. 1–40, 2005.

[32] L. A. Zadeh, "Is there a need for fuzzy logic?" *Information Sciences*, vol. 178, no. 13, pp. 2751–2779, 2008.

[33] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications tomodeling and control," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 15, no. 1, pp. 116–132, 1985.

[34] M. Sugeno, *Fuzzy Control*, North-Holland Publishing, 1988.

[35] Y. Tang and X. Yang, "Symmetric implicational method of fuzzy reasoning," *International Journal of Approximate Reasoning*, vol. 54, no. 8, pp. 1034–1048, 2013.

[36] M. Luo and N. Yao, "Triple I algorithms based on Schweizer-sklar operators in fuzzy reasoning," *International Journal of Approximate Reasoning*, vol. 54, no. 5, pp. 640–652, 2013.

[37] L. Zhang and K.-Y. Cai, "Optimal fuzzy reasoning and its robustness analysis," *International Journal of Intelligent Systems*, vol. 19, no. 11, pp. 1033–1049, 2004.

[38] Z. Q. Wu, M. Masaharu, and Y. Shi, "An improvement to Kóczy and Hirota's interpolative reasoning in sparse fuzzy rule bases," *International Journal of Approximate Reasoning*, vol. 15, no. 3, pp. 185–201, 1996.

[39] L. A. Zadeh, "Computing with words and perceptions—a paradigm shift," in *Proceedings of the IEEE International Conference on Information Reuse and Integration (IRI '09)*, pp. 450–452, Las Vegas, Nev, USA, August 2009.

[40] J. Casasnovas and J. V. Riera, "On the addition of discrete fuzzy numbers," *WSEAS Transactions on Mathematics*, vol. 5, no. 5, pp. 549–554, 2006.

[41] J. Casasnovas and J. V. Riera, "Weighted means of subjective evaluations," in *Soft Computing and Humanities in Social Sciences*, R. Seizing and V. Sanz, Eds., vol. 273 of *Studies in Fuzziness and Soft Computing*, pp. 323–345, Springer, Berlin, Germany, 2012.

[42] W. Voxman, "Canonical representations of discrete fuzzy numbers," *Fuzzy Sets and Systems*, vol. 118, no. 3, pp. 457–466, 2001.

[43] G. Wang, C. Wu, and C. Zhao, "Representation and operations of discrete fuzzy numbers," *Southeast Asian Bulletin of Mathematics*, vol. 28, pp. 1003–1010, 2005.

[44] M. Charles, J. Grinstead, and L. Snell, *Introduction to Probability*, American Mathematical Society, 1997.

[45] R. C. Williamson and T. Downs, "Probabilistic arithmetic. I. Numerical methods for calculating convolutions and dependency bounds," *International Journal of Approximate Reasoning*, vol. 4, no. 2, pp. 89–158, 1990.

[46] L. A. Zadeh, "Probability measures of fuzzy events," *Journal of Mathematical Analysis and Applications*, vol. 23, no. 2, pp. 421–427, 1968.

[47] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets, Theory and Applications*, World Scientific, Singapore, 1994.

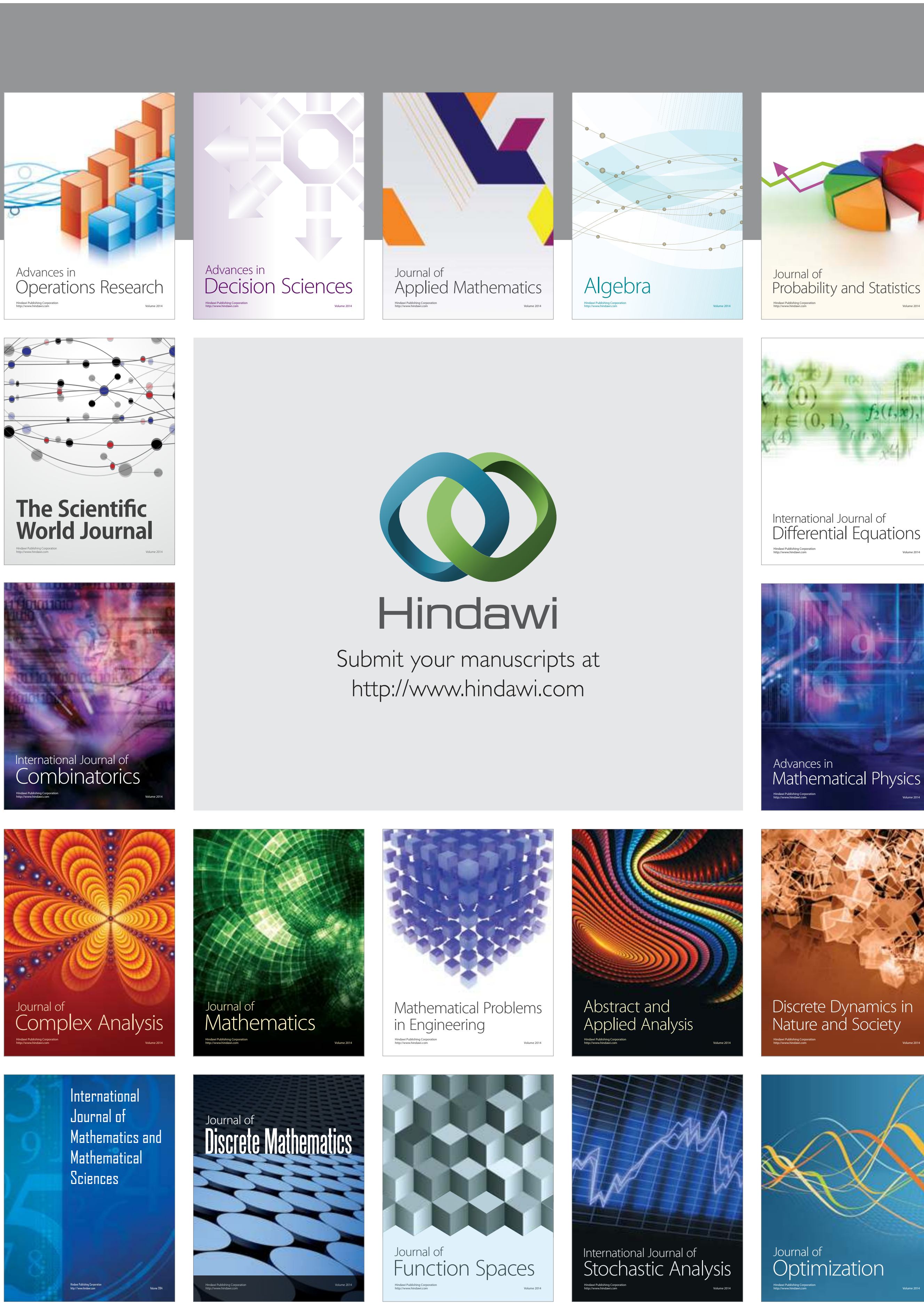
[48] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets, Theory and Applications*, World Scientific, River Edge, NJ, USA, 1994.

[49] L. T. Kóczy and K. Hirota, "Rule interpolation by α-level sets in fuzzy approximate reasoning," *Bulletin for Studies and Exchanges on Fuzziness and Its Applications*, vol. 46, pp. 115–123, 1991.

[50] R. Aliev and K. Memmedova, "Application of Z-number based modeling in psychological research," *Computational Intelligence and Neuroscience*, vol. 2015, Article ID 760403, 7 pages, 2015.

[51] L. A. Zadeh, "Z-numbers—a new direction in the analysis of uncertain and complex systems," in *Proceedings of the 7th IEEE International Conference on Digital Ecosystems and Technologies (IEEE-DEST '13)*, July 2013.

[52] K. Mammadova, "Modeling of impact of pilates on students performance under Z-information," in *Proceedings of the 8th World Conference on Intelligent Systems for Industrial Automation (WCIS '14)*, pp. 171–178, Tashkent, Uzbekistan, 2014.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