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Abstract

The Z-splines are moment conserving cardinal splines of compact support. They are constructed using Hermite-Birkhoff curves that reproduce explicit finite difference operators computed by Taylor series expansions. These curves are unique. The Z-splines are explicit piecewise polynomial interpolation kernels of cumulative regularity and accuracy. They are succesive spline approximations to the perfect reconstruction filter *sinc(x)*. It is found that their interpolation properties: quality, regularity, approximation order and discrete moment conservation, are related to a single basic concept: the exact representation of polynomials by a long enough Taylor series expansion.

Keywords: interpolation, splines, approximation, moment, conservation, piecewise polynomials, Vandermonde matrix, finite differences, Hermite interpolation

Subject Classification: 65D05, 65D07

1 Introduction

Let m be a positive integer and $f = f(x) \in C^{2m-1}[a,b]$ be a band-limited function whose value is known at $n+1$ given point values $y_j = f(x_j)$ $(j = 0, 1, 2, \ldots, n)$, such that $x_0 = a$ and $x_n = b$.

General cardinal spline interpolation of compact support consists in constructing the piecewise polynomial basis functions $\tilde{Z}_{m,j}(x)$ that produce interpolations of the form

$$f_m(x) = \sum_{j=0}^{n} y_j \tilde{Z}_{m,j}(x - x_j), \tag{1}$$

such that $f_m \in C^{m-1}[a,b]$, $f_m(x_j) = f(x_j)$ and $\tilde{Z}_{m,j}(x - x_j) = 0$ for $x \geq x_{j+m}$ or $x \leq x_{j-m}$.

If the data points are given at the doubly-infinite sequence of integers $x_j = j$ for $j \in \mathbb{Z}$, the problem is known as the Cardinal Interpolation Problem. The explicit solution for the case of an infinitely smooth function, known as the cardinal series

$$f(x) = \sum_{j=-\infty}^{\infty} y_j \frac{\sin \pi (x-j)}{\pi (x-j)}, \tag{2}$$

is the exact and unique solution for the case of an infinitely smooth band-limited function given at equidistant points. Whittaker (1915) reported its discovery in [10], although it has been given as early as 1899 by Borel. The Shannon-Whittaker theory tells us that the reconstruction is perfect given a function $f(x)$ whose bandwidth is limited to the finite interval $k \in [-\pi, \pi]$.

Schoenberg's theory of cardinal splines (1946) is the first generalization of (2). He solved the problem using an implicit formulation by solving a linear system of equations based on the so-called B-splines [7]. He was able to study an explicit form of the equivalent cardinal splines and found that the kernels that he was using for higher than first-order interpolation were of infinite support, creating some problems for general interpolation over finite domains. This problem had to be solved by the introduction of artificial boundary conditions.

Cardinal splines of compact support are known for $m = 1$ and 2, in the form of the "hat function" and the Catmull-Rom spline [3], also called the cubic convolution kernel and the modified or cardinal cubic spline. Splines of higher regularity are known as reported in [6] but all of them have the same order of accuracy as the cubic convolution kernel.

In this paper it is shown how to couple Taylor finite differences operators and Hermite-Birkhoff interpolation, also known as *fully* Hermite interpolation, such that they produce a compactly supported piecewise polynomial basis of cumulative moment conservation, accuracy and regularity. The basis is derived in *Section 2* for a doubly-infinite sequence of equidistant data points. The interpolation properties of the equidistant basis are proved using the theory of convolution based approximations of Strang and Fix [8] and adding a vanishing moments property not described in that theory.

In *Section 3* it is shown how to generalize this result to arbitrarily spaced data points and bounded domains.

2 Compactly Supported Cardinal Spline Interpolation

If $x_j = j$ is a doubly-infinite sequence of integers $j \in \mathbb{Z}$, then the general explicit interpolation formula of (1) can be written in a basis made of shifts of a single function Z_m as follows

$$f_m(x) = \sum_{j=-\infty}^{\infty} y_j Z_m(x-j)\,. \tag{3}$$

The construction of the basis Z_m is based on the inversion of Taylor series expansions, given by the well known Taylor finite differences, and the construction of Hermite-Birkhoff piecewise polynomial curves.

2.1 The Finite Differences Matrix

Define the vector of discrete function values

$$F_m = [f(-m+1), f(-m+2), ..., f(0), ..., f(m-2), f(m-1)]^T, \tag{4}$$

and the vector of derivatives at the origin

$$F'_m = [f(0), f'(0), f''(0), ..., f^{(2m-2)}(0)]^T, \tag{5}$$

such that the “chopped” Maclaurin (or Taylor) series expansions can be written in matrix form as

$$F_m = V_m D_m F'_m, \tag{6}$$

where D_m is a diagonal matrix with entries $1/(l-1)!$ for $l = 1, 2, \ldots, 2m\text{-}1$, and V_m is a Vandermonde matrix whose (l,p) entry is $(-(m-1)+(l-1))^{p-1}$.

The finite differences matrix A_m of dimensions $(2m-1)\times(2m-1)$ is defined as the inverse of the Taylor series expansion matrix $V_m D_m$, so that (6) implies

$$F'_m = A_m F_m, \quad A_m = D_m^{-1} V_m^{-1}. \tag{7}$$

D_m^{-1} is a diagonal matrix with entries $(l-1)!$ and the analytic form of the inverse of the Vandermonde matrix is known from the work of Spitzbart and Macon [5], recently written in a very interesting form by Bender *et. al.* in [1]. In general, if the (l,p) element of the Vandermonde matrix is given as s_l^{p-1}, for p and $l = 1, 2, \ldots, 2m\text{-}1$, the inverse of the Vandermonde matrix can be constructed as

$$[V]^{-1}_{l,p} = \frac{(-1)^{2m-1-l}}{(2m-1-l)!(l-1)!} \quad v_{l,p}, \tag{8}$$

where $v_{l,p}$ is the coefficient of x^{l-1} in the polynomial

$$\frac{(x-s_1)(x-s_2)(x-s_3)\ldots(x-s_{2m-1})}{x-s_p}. \tag{9}$$

Examples:

$A_1 = [1]$,

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \\ 1 & -2 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{12} & \frac{-2}{3} & 0 & \frac{2}{3} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{4}{3} & \frac{-5}{2} & \frac{4}{3} & \frac{-1}{12} \\ \frac{-1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{-1}{60} & \frac{3}{20} & \frac{-3}{4} & 0 & \frac{3}{4} & \frac{-3}{20} & \frac{1}{60} \\ \frac{1}{90} & \frac{-3}{20} & \frac{3}{2} & \frac{-49}{18} & \frac{3}{2} & \frac{-3}{20} & \frac{1}{90} \\ \frac{1}{8} & -1 & \frac{13}{8} & 0 & \frac{-13}{8} & 1 & \frac{-1}{8} \\ \frac{-1}{6} & 2 & \frac{-13}{2} & \frac{28}{3} & \frac{-13}{2} & 2 & \frac{-1}{6} \\ \frac{-1}{2} & 2 & \frac{-5}{2} & 0 & \frac{5}{2} & -2 & \frac{1}{2} \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix}.$$

2.2 The Z-splines

Definition 1 (The Z-splines) *For $m = 1, 2, 3, \ldots$, the Z-splines $Z_m = Z_m(x)$ are defined as the C^{m-1} Hermite-Birkhoff curves (piecewise polynomials of degree 2m-1) that reproduce the finite differences operators given by the m first rows of the finite differences matrix A_m.*

More precisely, the finite differences matrix generates the approximations to the derivatives of f as

$$\frac{d^{(p-1)}}{dx} f(x)|_{x=0} \approx \sum_{j=-m+1}^{m-1} y_j [A_m]_{p,j+m}, \quad p = 1, 2, \ldots, 2m-1, \tag{10}$$

and the Z-spline derivatives, up to the *(m-1)*-th, are

$$\frac{d^{(p-1)}}{dx} f(x)|_{x=0} \approx \sum_{j=-m+1}^{m-1} y_j Z_m^{(p-1)}(-j), \quad p = 1, 2, \ldots, m, \tag{11}$$

so the Z-spline and the finite differences matrix must match coefficients for $1 \leq p \leq m$ as follows

$$Z_m^{(p-1)}(-j) = [A_m]_{p,j+m}. \tag{12}$$

The solution to this problem using Hermite-Birkhoff piecewise polynomials of degree $2m-1$ exists and is unique. In every interval we are given $2m$ conditions over the function value and its derivatives that we are matching using a $2m-1$ degree polynomial with $2m$ coefficients. Polynomials of lower degree will not be able to match all the conditions and polynomials of higher degree will produce a larger computational complexity.

In general, the solution to the problem of constructing the piecewise polynomial given the value of the function and its first $m-1$ derivatives at the extremes of every subinterval $[x_j, x_{j+1}]$, is known as the *fully* Hermite or Hermite-Birkhoff interpolation formula [2].

The cardinal Z-splines, Hermite-Birkhoff piecewise polynomials where the derivatives of the function match the Taylor finite differences coefficients, are constructed for $x \in [j, j+1]$ as

$$Z_m(x) = \sum_{p=0}^{m-1} \left(Z_m^{(p)}(j) B_{p0}(x) + Z_m^{(p)}(j+1) B_{p1}(x) \right), \tag{13}$$

where

$$B_{p0}(x) = \frac{1}{p!}(x-j)^p \left(\sum_{\nu=0}^{m-p-1} (x-j)^\nu b_{\nu 0} \right) ((j+1)-x)^m, \tag{14}$$

$$B_{p1}(x) = \frac{1}{p!}(x-(j+1))^p \left(\sum_{\nu=0}^{m-p-1} (x-(j+1))^\nu b_{\nu 1} \right) (x-j)^m, \tag{15}$$

$$b_{\nu 0} = \frac{1}{\nu!} \frac{d^\nu}{dx^\nu} \left(\frac{1}{((j+1)-x)^m} \right)_{x=j}, \tag{16}$$

$$b_{\nu 1} = \frac{1}{\nu!} \frac{d^\nu}{dx^\nu} \left(\frac{1}{(x-j)^m} \right)_{x=j+1}. \tag{17}$$

Note that the difference between the so-called Hermite splines, where the derivatives are assumed to be given, and the Z-splines is the built-in high-order accurate $(m-1)$-th continuous derivatives of the latter.

Examples:

$$Z_1(x) = \begin{cases} 1-|x| & |x| \le 1, \\ 0 & |x| > 1. \end{cases} \tag{18}$$

$$Z_2(x) = \begin{cases} 1 - \frac{5}{2}x^2 + \frac{3}{2}|x|^3 & |x| \le 1, \\ \frac{1}{2}(2-|x|)^2(1-|x|) & 1 \le |x| \le 2, \\ 0 & |x| > 2. \end{cases} \tag{19}$$

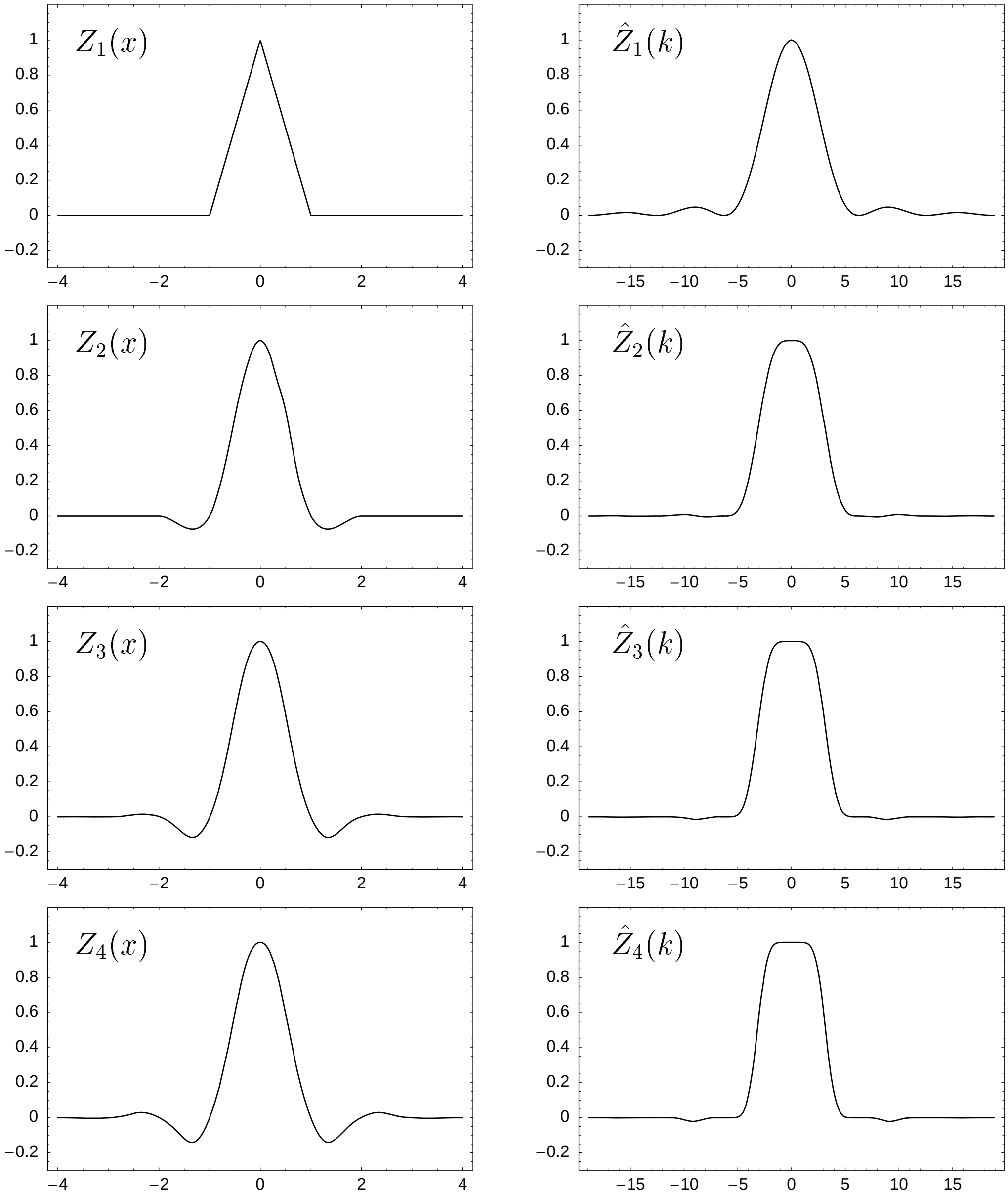

Fig. 1. The first four cardinal Z-splines Z_m and their Fourier transforms $\hat{Z}_m$.

$$Z_3(x) = \begin{cases} 1 - \frac{15}{12}x^2 - \frac{35}{12}|x|^3 + \frac{63}{12}x^4 - \frac{25}{12}|x|^5 & |x| \leq 1, \\ -4 + \frac{75}{4}|x| - \frac{245}{8}x^2 & \\ \qquad + \frac{545}{24}|x|^3 - \frac{63}{8}x^4 + \frac{25}{24}|x|^5 & 1 \leq |x| \leq 2, \\ 18 - \frac{153}{4}|x| + \frac{255}{8}x^2 & \\ \qquad - \frac{313}{24}|x|^3 + \frac{21}{8}x^4 - \frac{5}{24}|x|^5 & 2 \leq |x| \leq 3, \\ 0 & |x| > 3. \end{cases} \tag{20}$$

$$
Z_4(x) = \begin{cases}
1 - \frac{49}{36}x^2 - \frac{959}{144}x^4 + \frac{2569}{144}|x|^5 - \frac{727}{48}x^6 + \frac{623}{144}|x|^7 & |x| \leq 1, \\
\frac{138}{5} - \frac{8617}{60}|x| + \frac{12873}{40}x^2 - \frac{791}{2}|x|^3 & \\
\qquad + \frac{4557}{16}x^4 - \frac{9583}{80}|x|^5 + \frac{2181}{80}x^6 - \frac{623}{240}|x|^7 & 1 \leq |x| \leq 2, \\
-440 + \frac{25949}{20}|x| - \frac{117131}{72}x^2 + \frac{2247}{2}|x|^3 & \\
\qquad - \frac{66437}{144}x^4 + \frac{81109}{720}|x|^5 - \frac{727}{48}x^6 + \frac{623}{720}|x|^7 & 2 \leq |x| \leq 3, \\
\frac{3632}{5} - \frac{7456}{5}|x| + \frac{58786}{45}x^2 - 633|x|^3 & \\
\qquad + \frac{26383}{144}x^4 - \frac{22807}{720}|x|^5 + \frac{727}{240}x^6 - \frac{89}{720}|x|^7 & 3 \leq |x| \leq 4, \\
0 & |x| > 4.
\end{cases} \tag{21}
$$

2.3 The Properties of the Z-splines

Theorem 2 (Properties of the Z-splines) *The equidistant Z-splines Z_m, defined in the previous section, have the following properties:*

1. *They are unique.*

2. *They are orthogonal to their translates in the discrete space of the data points.*

3. *They have minimum compact support.*

4. *They are an exact basis for the monomials x, x^2, x^3, ..., x^{2m-2} and their linear combinations.*

5. *They conserve the first $2m-1$ discrete moments.*

6. *Their Fourier transforms are unity at zero and have zeroes of order $2m-1$ at the multiples of 2π.*

7. *For sufficiently smooth functions they are accurate of order $2m-1$.*

8. *They converge to the cardinal series kernel sinc(x).*

Proof: The cardinal Z-splines are unique because they are the solution to a unique Hermite-Birkhoff interpolation problem. There are $2m$ conditions for the function and its $(m-1)-$th first derivatives at consecutive nodes, and $2m$ coefficients in the $(2m-1)-$th degree Hermite-Birkhoff polynomial. They are orthogonal to their translates in the space of the discrete data points because

(12) for $n = 1$ means

$$Z_m(j) = \delta_{0j}. \tag{22}$$

They have minimum compact support because the finite differences generated by the matrix A_m have minimum compact support by construction. The kernel has $m-1$ vanishing derivatives at the most external nodes such that the Hermite-Birkhoff curve

$$Z_m \equiv 0 \text{ for } |x| > m. \tag{23}$$

The Taylor series expansion is exact for all the functions for which the $(2m-1)$-th derivative and the higher ones are zero. Therefore the finite differences matrix A_m produces exact derivatives for all the polynomials up to degree $2m-2$. The Hermite-Birkhoff spline generated with the exact derivatives represents exactly those polynomials and therefore the Z-splines represent exactly polynomials up to degree $2m-2$. More precisely, for $n \leq 2m-2$,

$$\sum_{j=-\infty}^{\infty} j^n Z_m(x-j) = x^n. \tag{24}$$

In their theory of convolution base approximations, Strang and Fix [8] have proven that the Properties **4** to **7** are equivalent. The difference of our proof with the theory of Strang and Fix is that the Z-splines have vanishing moments and not only constant. More precisely, it is not difficult to prove that (24) implies that for $n \leq 2m-2$

$$\sum_{j=-\infty}^{\infty} (x-j)^n Z_m(x-j) = \delta_{0n}. \tag{25}$$

Integrating (25) and using the orthogonality of the translates (22) implies that

$$\int_{-m}^{m} x^n Z_m(x) dx = \delta_{0n}, \tag{26}$$

which expresses orthonormality of the Z-splines with respect to the monomials, if we define the integral as the inner product.

The interpolation error for equidistant data taken at intervals $\Delta x = x_{j+1} - x_j$ is

$$f(x) - f_m(x) = f(x) - \sum_{j=-\infty}^{\infty} f(x_j) Z_m\left(\frac{x - x_j}{\Delta x}\right). \tag{27}$$

Using (24) for $n = 0$ and substituting $f(x_j)$ by its Taylor expansion leads to

$$f(x) - f_m(x) = \sum_{j=-\infty}^{\infty} \left(\sum_{p=1}^{2m-2} \frac{f^{(p)}(x)}{p!}(x - x_j)^p + O(\Delta x^{2m-1}) \right) Z_m\left(\frac{x - x_j}{\Delta x}\right), \tag{28}$$

Equation (25) for Δx arbitrary and $p \leq 2m - 2$ reads

$$\sum_{j=-\infty}^{\infty} (x - x_j)^p Z_m\left(\frac{x - x_j}{\Delta x}\right) = \delta_{0p}, \tag{29}$$

and therefore (28) proves that the interpolation error has the following bound as the data interval goes to zero:

$$||f - f_m||_{L_2} \leq C(\Delta x)^{2m-1} ||f^{(2m-1)}||_{L_2}, \tag{30}$$

where C is a real constant independent of f.

The Fourier transform of the interpolation kernel and its derivatives is

$$\hat{Z_m}^{(n)}(k) = \int_{-m}^{m} (-ix)^n Z_m(x) e^{-ikx} dx. \tag{31}$$

From this expression and given that the kernel Z_m is orthogonal to the first $2m - 1$ monomials, the symmetries that lead to (26) are not altered by the Fourier transform if $k = 2\pi l$ with l integer. More precisely, for $n \leq 2m - 2$,

$$\begin{aligned} \hat{Z_m}^{(n)}(0) &= \delta_{0n}, \\ \hat{Z_m}^{(n)}(2\pi l) &= 0, \quad l \in \mathbb{Z}, \quad l \neq 0. \end{aligned} \tag{32}$$

This is equivalent to proving that the Z-splines converge to the infinite support *sinc* kernel that has infinite-order zeroes in Fourier space. Another way to

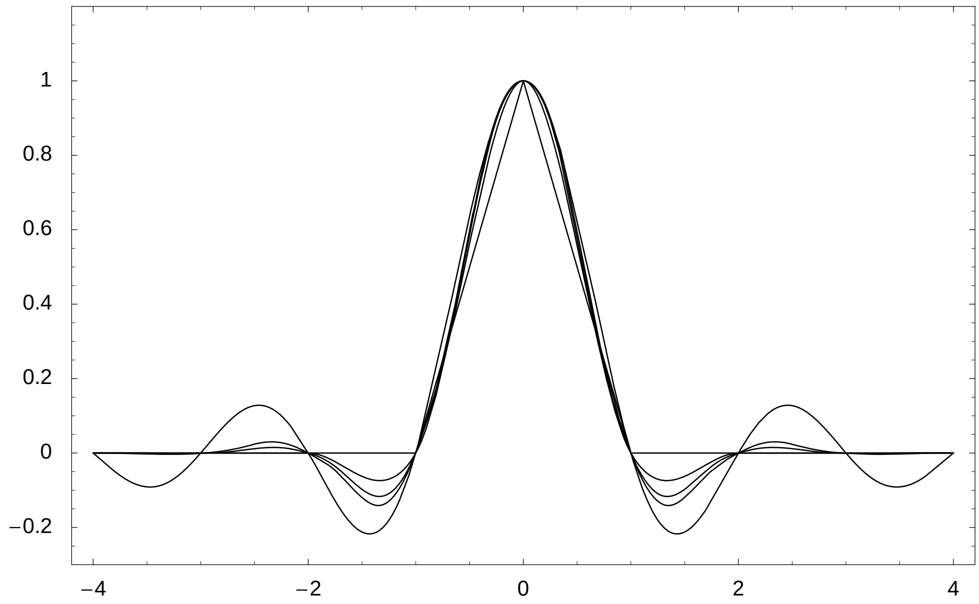

Fig. 2. The first four cardinal Z-splines and the *sinc* function.

understand the convergence to the *sinc* function is to notice that the *sinc* kernel produces a unique and infinite set of "finite" differences coefficients

$$\frac{d^{(n)}}{dx}f(0) = \sum_{j=-\infty}^{\infty} f(j)\frac{d^{(n)}}{dx}\left(\frac{sin\pi(x-j)}{\pi(x-j)}\right)_{x=0}, \tag{33}$$

to which the coefficients of the finite differences built in the Z-splines converge. □

The properties of the kernels have also been observed numerically by computing the vanishing moments property for any value of $x \in [0,1)$. Figure 3 is a graphic representation of the theory of the Z-splines. It shows a comparison of the first four cardinal Z-splines and the *sinc* function.

3 Z-splines for Arbitrarily Spaced Data and Bounded Domains

The cardinal Z-splines can be generalized to be used for data given at general x_j's such that $x_{j+1} > x_j$ and $j = 0,\ 1,\ 2,\ \ldots,\ n.$

The general Z-splines $\tilde{Z}_{m,j}$ can be studied in two cases given by

$$\tilde{Z}_{m,j}(x) = \begin{cases} \tilde{Z}_m(x) & \text{for } j > m \text{ and } n-j > m, \\ \tilde{Z}_{m,j}(x) & \text{for } j \le m \text{ or } \quad n-j \le m, \end{cases} \tag{34}$$

where $\tilde{Z}_m$ is the arbitrarily spaced, central Z-spline and $\tilde{Z}_{m,j}$ is the arbitrarily spaced, one-sided Z-spline. For convenience, the equidistant, one-sided Z-splines are denoted by $Z_{m,j}$.

The construction of the general Z-splines is based on the generalized finite differences matrix $A_{m,j}$, defined in analogy to the cardinal interpolation case.

Define the vector of discrete function values

$$F_{m,j} = [f(x_{i+j-m+1}), f(x_{i+j-m+2}), ..., f(x_{i+j}), ..., f(x_{i+j+m-1})]^T, \qquad (35)$$

where i=0 when $j > m$ or n-$j > m$, i = (m-1)-j when $j \leq m$ and i = n-j-(m-1) when n-$j \leq m$, and the vector of derivatives at the node x_j

$$F'_{m,j} = [f(x_j), f'(x_j), f''(x_j), ..., f^{(2m-2)}(x_j)]^T, \qquad (36)$$

so that the matrix form of the Taylor series expansions is

$$F_{m,j} = V_{m,j} D_m F'_{m,j}, \qquad (37)$$

where D_m is a diagonal matrix with entries $1/(l-1)!$ for l = 1, 2, ..., 2m-1, and $V_{m,j}$ is a Vandermonde matrix whose (l,p) entry is $(x_{i+j-(m-1)+(l-1)} - x_j)^{p-1}$.

The arbitrarily spaced finite differences matrix is defined as

$$A_{m,j} = D_m^{-1} V_{m,j}^{-1}. \qquad (38)$$

The inverse of the Vandermonde matrix has been given before in general form in (8).

Definition 3 (The Generalized Z-splines) *For m = 1, 2, 3,..., the generalized Z-splines $\tilde{Z}_{m,j} = \tilde{Z}_{m,j}(x)$ are defined as the C^{m-1} Hermite-Birkhoff curves (piecewise polynomials of degree 2m-1) that reproduce the finite differences operators given by the m first rows of the arbitrarily spaced finite differences matrices $A_{m,j}$.*

More precisely, as in the cardinal interpolation case, the general finite differences matrix generates the approximations to the derivatives of the function f at $x = x_j$ as

$$\frac{d^{(p-1)}}{dx} f(x)|_{x=x_j} \approx \sum_{l=i+j-(m-1)}^{i+j+(m-1)} y_l [A_{m,j}]_{p,l+m-i-j}, \qquad (39)$$

and the derivatives using the general Z-splines are

$$\frac{d^{(p-1)}}{dx} f(x)|_{x=x_j} \approx \sum_{l=i+j-(m-1)}^{i+j+(m-1)} y_l \tilde{Z}_{m,l}^{(p-1)}(x_j - x_l), \tag{40}$$

so the general Z-splines and the general finite differences matrices must match coefficients for $1 \le p \le m$

$$\tilde{Z}_{m,l}^{(p-1)}(x_j - x_l) = [A_{m,j}]_{p,l+m-i-j}. \tag{41}$$

Given the finite differences coefficients in (41), then the arbitrarily spaced Z-splines are constructed for every interval $x \in [x_j, x_{j+1}]$ as

$$\tilde{Z}_{m,i}(x) = \sum_{p=0}^{m-1} \left(\tilde{Z}_{m,i}^{(p)}(x_j) B_{p0}(x) + \tilde{Z}_{m,i}^{(p)}(x_{j+1}) B_{p1}(x) \right), \tag{42}$$

where

$$B_{p0}(x) = \frac{1}{p!}(x - x_j)^p \left(\sum_{\nu=0}^{m-p-1} (x - x_j)^\nu b_{\nu 0} \right) l_0^m(x), \tag{43}$$

$$B_{p1}(x) = \frac{1}{p!}(x - x_{j+1})^p \left(\sum_{\nu=0}^{m-p-1} (x - x_{j+1})^\nu b_{\nu 1} \right) l_1^m(x), \tag{44}$$

$$b_{\nu k} = \frac{1}{\nu!} \left(\frac{1}{l_k^m(x)} \right)^{(\nu)}_{x=x_{j+k}}, \; l_0(x) = \frac{x - x_{j+1}}{x_j - x_{j+1}}, \; l_1(x) = \frac{x - x_j}{x_{j+1} - x_j}. \tag{45}$$

As an example, the centered cubic Z-spline for arbitrary intervals a_1, a_2, a_3

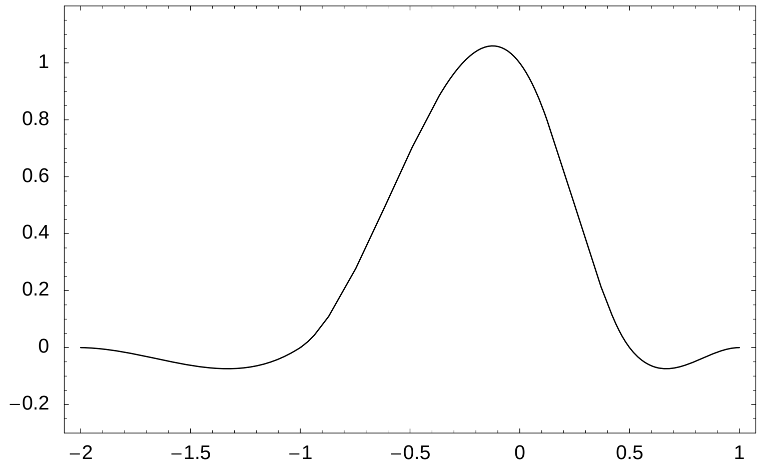

Fig. 3. The arbitrarily spaced cubic Z-spline (46) for $a_1 = a_2 = 1$ and $a_3 = a_4 = 0.5$.

and a_4 is given by:

$$\tilde{Z}_2(x) = \begin{cases} 0 & x < -a_1 - a_2 \\ \left(\frac{a_2+a_1}{a_1}\right) + \left(\frac{3a_2+a_1}{a_2 a_1}\right) x & \\ \quad + \frac{3a_2+2a_1}{a_2 a_1 (a_2+a_1)} x^2 + \frac{x^3}{a_2 a_1 (a_2+a_1)} & -a_1 - a_2 \leq x \leq -a_2, \\ 1 - \left(\frac{1}{a_3} - \frac{1}{a_2}\right) x & \\ \quad - \frac{a_3+2(a_2+a_1)}{a_3 a_2 (a_2+a_1)} x^2 - \frac{a_3+a_2+a_1}{a_3 a_2^2 (a_2+a_1)} x^3 & -a_2 \leq x \leq 0, \\ 1 + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) x & \\ \quad - \frac{a_2+2(a_3+a_2)}{a_2 a_3 (a_3+a_4)} x^2 - \frac{a_2+a_3+a_4}{a_2 a_3^2 (a_3+a_4)} x^3 & 0 \leq x \leq a_3, \\ \left(\frac{a_3+a_4}{a_4}\right) - \left(\frac{3a_3+a_4}{a_3 a_4}\right) x & \\ \quad + \frac{3a_3+2a_4}{a_3 a_4 (a_3+a_4)} x^2 - \frac{x^3}{a_3 a_4 (a_3+a_4)} & a_3 \leq x \leq a_3 + a_4, \\ 0 & x > a_3 + a_4. \end{cases} \quad (46)$$

It is important to notice that $\tilde{Z}_2$ doesn't have the maximum at $x = 0$ as can be observed in Fig. 3. Therefore the arbitrarily spaced spline is not analogous to a continuous mapping of the centered Z_2, because in that case the maximum occurs at $x = 0$.

The following examples are the one-sided cubic cardinal Z-splines, the second index indicates the number of nodes that lie between the center of the kernel

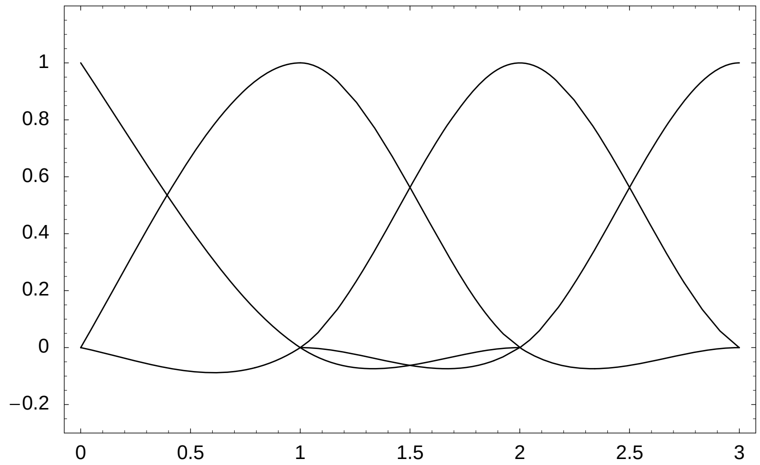

Fig. 4. One-sided cubic Z-splines.

and the left boundary:

$$Z_{2,0}(x) = \begin{cases} 1 - \frac{7}{6}x - \frac{1}{6}x^2 + \frac{1}{3}x^3 & 0 \le x \le 1, \\ Z_2(x) & x > 1. \end{cases} \tag{47}$$

$$Z_{2,1}(x) = \begin{cases} \frac{4}{3}x + \frac{1}{3}x^2 - \frac{2}{3}x^3 & 0 \le x \le 1, \\ Z_2(x-1) & x > 1. \end{cases} \tag{48}$$

$$Z_{2,2}(x) = \begin{cases} -\frac{1}{6}x - \frac{1}{6}x^2 + \frac{1}{3}x^3 & 0 \le x \le 1, \\ Z_2(x-2) & x > 1. \end{cases} \tag{49}$$

One-sided interpolations using Hermite-Birkhoff piecewise polynomials have been used by Higham [4] in developing the so-called highly continuous Runge-Kutta interpolants. The difference between the Runge-Kutta interpolants and the one-sided Z-splines is, as in the case of the Hermite splines, the built-in higher order continuity of the Z-splines.

Most of the properties of the equidistant, centered Z-splines can be extended to the arbitrarily spaced and one-sided Z-splines because they are also constructed inverting Taylor expansions and matching the coefficients of the finite differences stencils to Hermite-Birkhoff piecewise polynomials. Nevertheless, properties **5**, **6** and **8** cannot be extended in a straightforward manner because of the broken symmetry for arbitrary intervals and near boundaries.

4 Conclusions

A unique family of moment conserving cardinal splines of compact support has been found. Its elements have been named Z-splines and they have been generalized to arbitrary-interval data and bounded domains. They are very useful computationally because of their minimum compact support, cumulative smoothness and discrete moment conservation. The main difference to Waring-Lagrange "classic" interpolation is that the Z-spline basis is constructed using finite differences derived from inversions of Taylor series expansions that are uniformly and absolutely convergent, while the "classic" interpolation formulas (including the B-splines) are based on divided differences that have problems for high degree. This work has described the method to obtain the complete family of Z-splines and it presents for the first time explicit formulas for the fifth- and seventh-order central Z-splines, as well as for the third-order one-sided and arbitrarily spaced Z-splines. The best explicit cardinal spline (or convolution spline) of compact support known up to this date was only third-order [6].

References

[1] C. M. Bender, D. C. Brody, B. K. Meister: Inverse of a Vandermonde Matrix, submitted to Journal of Physics A (2002)

[2] R. Bulirsch, H. Rutishauser: Interpolation und genäherte Quadratur, Mathematische Hilfsmittel des Ingenieurs (R. Sauer and I. Szabó, eds), Springer-Verlag, New York (1968)

[3] E. Catmull, R. Rom: A class of local interpolating splines, Computer Aided Geometric Design (R.E. Barnhill and R. F. Riesengeld, eds), Academic, New York, 317-326 (1974)

[4] D. J. Higham: Highly Continuous Runge-Kutta Interpolants, ACM Transactions on Mathematical Sofware **17**(3), 368-386 (1991)

[5] N. Macon, A. Spitzbart: Inverse of Vandermonde matrices, Amer. Math. Monthly **65**, 95-100 (1958).

[6] E. Meijering: A Chronology of Interpolation: From Ancient Astronomy to Modern Signal and Image Processing, Proceedings of the IEEE **90**(3), 319-342 (2002)

[7] I.J. Schoenberg: Cardinal Spline Interpolation, Philadelphia, PA: SIAM (1973)

[8] G. Strang, G. Fix: A Fourier analysis of the finite element variational method, Constructive Aspects of Functional Analysis, Edizioni Cremonese, Centro Internazionale Matematico Estivo, Rome, 793-840 (1973)

[9] B. Taylor: Methodus Incrementorum Directa and Inversa, London U.K. (1715)

[10] E. T. Whittaker: On the functions which are represented by the expansions of interpolation-theory, Proc. R. Soc. Edinburgh **35**, 181-194 (1915)