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Zagreb Polynomials of Three Graph Operators

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Abstract. In general, the relations among Zagreb polynomials on three graph operators are discussed in this paper. Specifically, relations between Zagreb polynomials of a graph G and a graph obtained by applying the operators $S(G)$, $R(G)$ and $Q(G)$ are investigated. In a separate section, the relation between Zagreb polynomial of a graph G and its corona is also described.

1. Introduction and Preliminaries

A topological index is a graph invariant applicable in chemistry. The first and second *Zagreb indices* are amongst the oldest and best known topological indices defined in 1972 by *Gutman* and are given with different names in the literature, such as the Zagreb group indices, the Zagreb group parameters and most often, the Zagreb indices. Zagreb indices were among the first indices introduced and has been used to study molecular complexity, chirality, ZE-isomerism and hetero-systems. Zagreb indices exhibited a potential applicability for deriving multi-linear regression models.

Let G be a connected graph with n vertices and m edges. The vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. For every vertex $v_i \in V(G)$, where $i = 1, 2, ..., n$, the *edge connecting* v_i and v_j is denoted by (v_i, v_j). Also the notation $d(v_i)$ denotes the *degree* of vertex v_i in G. There are two special graphs, namely *line graph* $L(G)$ and *subdivision graph* $S(G)$. In fact $L(G)$ is the graph whose vertices correspond to the edges of G such that two vertices adjacent if and only if the corresponding edges in G have a common vertex. Also $S(G)$ is the graph obtained from G by replacing each of its edge by a path of length two (or equivalently, by inserting an additional vertex into each edge of G).

The *first* and the *second Zagreb indices* (cf. [4]) are defined as

$$M_1(G) = \sum_{v_i \in V(G)} [d(v_i)^2] \quad \text{and} \quad M_2(G) = \sum_{(v_i, v_j) \in E(G)} [d(v_i).d(v_j)] .$$

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Considering the Zagreb indices, Fath-Tabar ([2]) defined *first* and the *second Zagreb polynomials* as

$$M_1(G,x) = \sum_{v_i,v_j \in E(G)} x^{d(v_i)+d(v_j)} \quad \text{and} \quad M_2(G,x) = \sum_{v_i,v_j \in E(G)} x^{d(v_i).d(v_j)},$$

respectively, where x is a variable. After that, in [3], the authors defined the *third Zagreb index*

$$M_3 = M_3(G) = \sum_{(v_i,v_j)\in E(G)} [|d(v_i) - d(v_j)|]$$

and *third Zagreb polynomials*

$$M_3(G,x) = \sum_{(v_i,v_j)\in E(G)} x^{|d(v_i)-d(v_j)|}.$$

In addition, in [10], Shuxian defined two polynomials related to the first Zagreb index as in the form

$$M_1^*(G,x) = \sum_{v_i \in V(G)} d(v_i).x^{d(v_i)} \quad \text{and} \quad M_0(G,x) = \sum_{v_i \in V(G)} x^{d(v_i)}$$

In graph theory, it is one of the important goal to define some (sharp) upper or lower bounds for a simple graph by considering the special indices and the graph products over them. In fact the Zagreb index (or coindex) is one of these indices that studied on it so much (see, for instance, [5, 6, 12] and the references cited in them). As we just indicated, although the Zagreb index itself has taken so much interest from graph theorist since last decades, in this paper, as a similar manner with known polynomials (given in the previous paragraph), we will mainly define the following Zagreb polynomials.

- $M_4(G,x) = \sum_{(v_i,v_j)\in E(G)} x^{d(v_i)[d(v_i)+d(v_j)]}$,
- $M_5(G,x) = \sum_{(v_i,v_j)\in E(G)} x^{d(v_j)[d(v_i)+d(v_j)]}$,
- $M_{a,b}(G,x) = \sum_{(v_i,v_j)\in E(G)} x^{[ad(v_i)+bd(v_j)]}$,
- $M'_{a,b}(G,x) = \sum_{(v_i,v_j)\in E(G)} x^{(d(v_i)+a)(d(v_j)+b)}$.

We just exhibit some relationships among these polynomials with some other known structures in here. However, we still believe that new application areas will (or can) be found for them.

We should also note that, other than $L(G)$ and $S(G)$, there exist two extra subdivision operators $R(G)$ and $Q(G)$ (cf. [11]) which will be needed in our results of this paper.

Definition 1.1 ([11]). *The operator $R(G)$ is defined as the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it. Moreover, the other operator $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining edges those pairs of these new vertices which lie on adjacent edges of G.*

Then we have the following lemma.

Lemma 1.2 ([11]). *One can re-write the subdivision operators described above as in the following:*

$$\begin{aligned} L(G) &:= (E(G), EE(G)), \\ S(G) &:= (V(G) \cup E(G), EV(G)), \\ R(G) &:= (V(G) \cup E(G), EV(G) \cup E(G)), \\ Q(G) &:= (V(G) \cup E(G), EV(G) \cup EE(G)). \end{aligned}$$

The organization of this paper is presented as follows: In Section 2, we investigate the relation between Zagreb polynomial of a graph G and a graph obtained by applying the operators $S(G)$, $R(G)$ and $Q(G)$. In Section 3, the relation between Zagreb polynomial of a graph G and its corona is described.

2. Relations Connecting Zagreb Polynomials on Operators $S(G)$, $R(G)$ and $Q(G)$

Let $v_1, v_2, v_3, \ldots, v_n$ be the n vertices of G and let $u_1, u_2, u_3, \ldots, u_m$ be the m subdivision vertices of G, where $n \geq 2$ and $m \geq 1$. In this part of the paper, we will present a relation connecting the first, second and third Zagreb polynomials of a connected graph G and its subdivision graph $S(G)$ and two graph operators $R(G)$ and $Q(G)$ (see Definition 1.1).

Theorem 2.1. *For the subdivision graph $S(G)$ of G, the Zagreb polynomials are given by*

$$M_1(S(G), x) = x^2[M_1^*(G, x)]\,,$$
$$M_2(S(G), x) = M_1^*(G, x^2)\,,$$
$$M_3(S(G), x) = \frac{1}{x^2}M_1^*(G, x)\,.$$

Proof. Let $u_1, u_2, \ldots, u_m$ be the m subdivision vertices on the m edges of $S(G)$. The degrees of all these subdivision vertices are 2 in $S(G)$. According to the definition, all edges having type (v_i, u_j) in $S(G)$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Therefore

$$M_1(S(G), x) = \sum_{(v_i,u_j)\in E(S(G))} x^{d(v_i)+d(u_j)} = \sum_{v_i\in V(G)} d(v_i).x^{d(v_i)+2} = x^2[M_1^*(G, x)]\,.$$

Similarly,

$$M_2(S(G), x) = \sum_{v_i\in V(G)} d(v_i)x^{2d(v_i)} = M_1^*(G, x^2)\,.$$

Further, for $M_3(S(G), x)$,

$$M_3(S(G), x) = \sum_{(v_i,u_j)\in E(S(G))} x^{|d(v_i)-d(u_j)|} = \sum_{v_i\in V(G)} d(v_i).x^{d(v_i)-2} = \frac{1}{x^2}M_1^*(G, x)\,.$$

Hence the result. □

Theorem 2.2. *For the graph $R(G)$, the Zagreb polynomials are given by*

$$M_1(R(G), x) = M_1(G, x^2) + x^2M_1^*(S(G), x^2)\,,$$
$$M_2(R(G), x) = M_2(G, x^4) + M_2(S(G), x^2)\,,$$
$$M_3(R(G), x) = M_3(G, x^2) + \frac{1}{x^2}M_1^*(S(G), x^2)\,.$$

Proof. By Definition 1.1, there are two types of edges in $R(G)$ which are either the edges of type (v_i, v_j) or the edges of type (v_i, u_j). Hence

$$\begin{aligned} M_1(R(G)) &= \sum_{(v_i,v_j)\in E(R(G))} x^{d(v_i)+d(v_j)} + \sum_{(v_i,u_j)\in E(R(G)} x^{d(v_i)+d(u_j)} \\ &= \sum_{(v_i,v_j)\in E(G)} x^{2(d(v_i)+d(v_j))} + \sum_{v_i\in V(S(G))} d(v_i)x^{2d(v_i)+2} \\ &= M_1(G, x^2) + x^2M_1^*(S(G), x^2)\,. \end{aligned}$$

Similarly,

$$\begin{aligned} M_2(R(G),x) &= \sum_{(v_i,v_j)\in E(R(G))} x^{4d(v_i)d(v_j)} + \sum_{(v_i,u_j)\in E(S(G)} x^{2d(v_i)d(u_j)} \\ &= M_2(G,x^4) + M_2(S(G),x^2)\,. \end{aligned}$$

Moreover, for $M_3(R(G),x)$,

$$\begin{aligned} M_3(R(G),x) &= \sum_{(v_i,v_j)\in E(R(G))} x^{|d(v_i)-d(v_j)|} + \sum_{(v_i,u_j)\in E(R(G)} x^{d(v_i)-d(u_j)} \\ &= \sum_{(v_i,v_j)\in E(G)} x^{2(d(v_i)-d(v_j))} + \sum_{v_i\in V(S(G))} d(v_i)x^{2d(v_i)-2} \\ &= M_3(G,x^2) + \frac{1}{x^2}M_1^*(S(G),x^2)\,, \end{aligned}$$

as required. □

Theorem 2.3. *For the graph $Q(G)$, the Zagreb polynomials are given by*

$$M_1(Q(G),x) = M_{2,1}(G,x) + M_{1,2}(G,x) + x^4 M_1(L(G),x)\,,$$

$$M_2(Q(G),x) = M_4(G,x) + M_5(G,x) + M'_{2,2}(L(G),x)\,,$$

$$M_3(Q(G),x) = M_1^*(G,x) + M_3(L(G),x)\,.$$

Proof. By Definition 1.1, there are two types of edges in $Q(G)$ which are either the edges of type (v_i,u_j) or the edges of type (u_j,u_k). Hence for all (v_i,u_j) and $(u_j,u_k)\in E(Q(G))$,

$$\begin{aligned} M_1(Q(G),x) &= \sum_{(v_i,u_j)\in E(Q(G))} x^{d(v_i)+d(u_j)} + \sum_{(u_j,u_k)\in E(Q(G))} x^{d(u_j)+d(u_k)} \\ &= \sum_{(v_i,v_j)\in E(G)} x^{2d(v_i)+d(v_j)} + \sum_{(v_i,v_j)\in E(G)} x^{d(v_i)+2d(v_j)} + \sum_{(u_j,u_k)\in E(L(G))} x^{d(u_j)+d(u_k)} \\ &= M_{2,1}(G,x) + M_{1,2}(G,x) + x^4 M_1(L(G),x)\,. \end{aligned}$$

In a similar way,

$$\begin{aligned} M_2(Q(G),x) &= \sum_{(v_i,u_j)\in E(Q(G))} x^{d(v_i).d(u_j)} + \sum_{(u_j,u_k)\in E(Q(G))} x^{d(u_j).d(u_k)} \\ &= \sum_{(v_i,v_j)\in E(G)} x^{d(v_i)(d(v_i)+d(v_j))} + \sum_{(v_i,v_j)\in E(G)} x^{d(v_j)(d(v_i)+d(v_j))} \\ &\quad + \sum_{(u_j,u_k)\in E(L(G))} x^{(d(u_j)+2)(d(u_k)+2)} \\ &= M_4(G,x) + M_5(G,x) + M'_{2,2}(L(G),x)\,. \end{aligned}$$

Finally, for $M_3(Q(G),x)$,

$$\begin{aligned} M_3(Q(G),x) &= \sum_{(v_i,u_j)\in E(Q(G))} x^{|d(v_i)-d(u_j)|} + \sum_{(u_j,u_k)\in E(Q(G))} x^{|d(u_j)-d(u_k)|} \\ &= \sum_{v_i\in V(G)} d(v_i)x^{d(v_i)} + \sum_{(u_j,u_k)\in E(L(G))} x^{|d(u_j)-d(u_k)|} \\ &= M_1^*(G,x) + M_3(L(G),x)\,. \end{aligned}$$

These end up the proof. □

3. Zagreb Polynomials on Corona of G

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and m edges. The *corona* of the graph G, denoted by G^+, is the graph obtained from G by adding n new vertices $v_1', v_2', \ldots, v_n'$ and joining vertices v_i' to v_i by an edge, where $i = 1, 2, \ldots, n$.

In this final section, we provide the relation connecting a connected graph G and its corona G^+ in terms of Zagreb polynomials.

Theorem 3.1. *For G^+, the first, second and third Zagreb polynomials are given by*

$$M_1(G^+,x) = x^2[M_1(G,x) + M_0(G,x)]\,,$$

$$M_2(G^+,x) = M_{1,1}(G,x) + xM_0(G,x)\,,$$

$$M_3(G^+,x) = M_3(G,x) + M_0(G,x)\,.$$

Proof. The definition of In G^+ implies that the degree of the vertices $v_1, v_2, \ldots, v_n$ (for all $i = 1, 2, \ldots, n$) are described by $d_{G^+}(v_i) = d_G(v_i) + 1$ while the degree of the vertices $v_1', v_2', \ldots, v_n'$ are given by $d_{G^+}(v_i') = 1$. There are actually two types of edges in G^+ that are the edges of type (v_i, v_j) and the edges of type (v_i, v_i'). Hence, for all types of the edges in G^+,

$$\begin{aligned} M_1(G^+,x) &= \sum_{(v_i,v_j)\in E(G^+)} x^{d(v_i)+d(v_j)} + \sum_{(v_i,v_{i'})\in E(G^+)} x^{d(v_i)+d(v_i')} \\ &= \sum_{(v_i,v_j)\in E(G)} x^{d(v_i)+d(v_j)+2} + \sum_{v_i\in V(G)} x^{d(v_i)+2} \\ &= x^2[M_1(G,x) + M_0(G,x)]\,. \end{aligned}$$

Similarly,

$$\begin{aligned} M_2(G^+,x) &= \sum_{(v_i,v_j)\in E(G^+)} x^{d(v_i)*d(v_j)} + \sum_{(v_i,v_{i'})\in E(G^+)} x^{d(v_i)*d(v_i')} \\ &= \sum_{(v_i,v_j)\in E(G)} x^{(d(v_i)+1)(d(v_j)+1)} + \sum_{v_i\in V(G)} x^{d(v_i)+1} \\ &= M_{1,1}(G,x) + xM_0(G,x)\,. \end{aligned}$$

For all edges in G^+,

$$\begin{aligned} M_3(G^+,x) &= \sum_{(v_i,v_j)\epsilon E(G^+)} x^{|d(v_i)-d(v_j)|} + \sum_{(v_i,v_{i'})\epsilon E(G^+)} x^{|d(v_i)-d(v_i')|} \\ &= \sum_{(v_i,v_j)\epsilon E(G)} x^{|d(v_i)-d(v_j)|} + \sum_{v_i\epsilon V(G)} x^{d(v_i)} \\ &= M_3(G,x) + M_0(G,x)\,. \end{aligned}$$

□

Theorem 3.2. *In the subdivision graph $S(G^+)$ of G^+, the Zagreb polynomials are given by*

$$M_1(S(G^+),x) = x^3[M_1^*(G,x) + M_0(G,x) + n]\,,$$

$$M_2(S(G^+),x) = x^2[M_1^*(G,x^2) + M_0(G,x^2) + n]\,,$$

$$M_3(S(G^+),x) = \frac{1}{x}[M_1^*(G,x) + nx^2]\,.$$

Proof. Again by the definition, $S(G^+)$ contains both all the vertices in G^+ and $m+n$ subdivision vertices having degree 2. Therefore, for all the edges in G^+, the first Zagreb polynomial is defined by

$$\begin{aligned} M_1(S(G^+),x) &= \sum_{v_i\in V(S(G^+))} x^{d(v_i)+2} + \sum_{v_{i'}\in V(S(G^+))} x^{d(v_i')+2} \\ &= \sum_{v_i\in V(G)} [d(v_i)+1]x^{d(v_i)+3} + nx^3 \\ &= x^3[M_1^*(G,x) + M_0(G,x) + n]\,. \end{aligned}$$

In the similar way,

$$\begin{aligned} M_2(S(G^+),x) &= \sum_{v_i\in V(G)} [d(v_i)+1]x^{2[d(v_i)+1]} + nx^2 \\ &= x^2[M_1^*(G,x^2) + M_0(G,x^2) + n]\,. \end{aligned}$$

For $M_3(S(G^+),x)$,

$$\begin{aligned} M_3(S(G^+),x) &= \sum_{(v_i,u_j)\in E(S(G^+))} x^{|d(v_i)-d(u_j)|} + \sum_{(v_{i'},u_j)\in E(S(G^+))} x^{|d(v_{i'})-d(u_j)|} \\ &\quad + \sum_{(v_i,u_{i'})\in E(S(G^+))} x^{|d(v_i)-d(u_{i'})|} \\ &= \sum_{v_i\in V(G)} [d(v_i) * x^{d(v_i)-1}] + nx \\ &= \frac{1}{x}[M_1^*(G,x) + nx^2]\,. \end{aligned}$$

Hence the result. □

Theorem 3.3. *For the operator $R(G^+)$, the Zagreb polynomials are given by*

$$M_1(R(G^+),x) = x^4[M_1(G,x^2) + M_1^*(G,x^2) + 2M_0(G,x^2) + n],$$

$$M_2(R(G^+),x) = M'_{1,1}(G,x^4) + x^4[2M_0(G,x^4) + M_1^*(G,x^4) + n],$$

$$M_3(R(G^+),x) = M_3(G,x^2) + 2M_0(G,x^2) + M_1^*(G,x^2) + n.$$

Proof.

$$\begin{aligned}
M_1(R(G^+),x) &= \sum_{(v_i,v_j)\in E(R(G^+))} x^{d(v_i)+d(v_j)} + \sum_{(v_i,v_{i'})\in E(R(G^+))} x^{d(v_i)+d(v_{i'})} \\
&\quad + \sum_{v_i\in V(R(G^+))} x^{d(v_i)+2} + \sum_{v_i\in V(R(G^+))} x^{d(v_{i'})+2} \\
&= \sum_{(v_i,v_j)\in E(G)} x^{2(d(v_i)+d(v_j))+4} + \sum_{v_i\in V(G)} x^{2(d(v_i+1)+2} \\
&\quad + \sum_{v_i\in V(G)} [d(v_i)+1]x^{2[d(v_i)+1]+2} + nx^4 \\
&= x^4\left[M_1(G,x^2) + M_1^*(G,x^2) + 2M_0(G,x^2) + n\right].
\end{aligned}$$

For $M_2(R(G^+),x)$,

$$\begin{aligned}
M_2(R(G^+),x) &= \sum_{(v_i,v_j)\in E(R(G^+))} x^{d(v_i).d(v_j)} + \sum_{(v_i,v_{i'})\in E(R(G^+))} x^{d(v_i).d(v_{i'})} \\
&\quad + \sum_{(v_i,u_j)\in E(R(G^+))} x^{2d(v_i)} + \sum_{(v_{i'},u_j)\in E(R(G^+))} x^{2d(u_{i'})} \\
&= \sum_{(v_i,v_j)\in E(G)} x^{4(d(v_i)+1).(d(v_j)+1)} + x^4\Bigg[2\sum_{v_i\in V(G)} x^{4(d(v_i))} \\
&\quad + \sum_{v_i\in V(G)} (d(v_i)x^{4.d(v_i)} + n\Bigg] \\
&= M'_{1,1}(G,x^4) + x^4\left[2M_0(G,x^4) + M_1^*(G,x^4) + n\right].
\end{aligned}$$

For $M_3(R(G^+),x)$,

$$\begin{aligned}
M_3(R(G^+),x) &= \sum_{(v_i,v_j)\in E(R(G^+))} x^{2|(d(v_i)-d(v_j))} + \sum_{v_i\in V(G)} x^{2d(v_i)} \\
&\quad + \sum_{v_i\in V(G)} (d(vi)+1)x^{2d(v_i)} + n \\
&= M_3(G,x^2) + 2M_0(G,x^2) + M_1^*(G,x^2) + n,
\end{aligned}$$

as required. □

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