Zero-Divisor Graph with Respect to an Ideal*

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Abstract

Let R be a commutative ring with nonzero identity and let I be an ideal of R. The zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set $\{x \in R \setminus I | \ xy \in I \text{ for some } y \in R \setminus I\}$ with distinct vertices x and y adjacent if and only if $xy \in I$. In the case $I = 0$, $\Gamma_0(R)$, denoted by $\Gamma(R)$, is the zero-divisor graph which has well known results in the literature. In this article we explore the relationship between $\Gamma_I(R) \cong \Gamma_J(S)$ and $\Gamma(R/I) \cong \Gamma(S/J)$. We also discuss when $\Gamma_I(R)$ is bipartite. Finally we give some results on the subgraphs and the parameters of $\Gamma_I(R)$.

Keywords: Zero-divisor graph, r-Partite graph, Clique number, Girth.

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1 Introduction and Preliminaries

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero divisors. The *zero-divisor graph*, $\Gamma(R)$, is the graph with vertices $Z(R)^* = Z(R)\setminus\{0\}$, the set of nonzero zero divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. In [7] Beck introduced the concept of a zero-divisor graph of a commutative ring. However, he lets all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. We adopt the approach used by D. F. Anderson and P. S. Livingston in [6] and consider only nonzero zero divisors as vertices of the graph. The zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g., [7, 6, 4, 10, 5, 1, 2].

In [11] Redmond introduced the definition of the zero-divisor graph with respect to an ideal. Let I be an ideal of R. The *zero-divisor graph of R with respect to I*, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set $\{x \in R \setminus I | \ xy \in I \text{ for some } y \in R \setminus I\}$ with distinct vertices x and y adjacent if and only if $xy \in I$. Thus if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$, and I is a nonzero prime ideal of R if and

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only if $\Gamma_I(R) = \emptyset$. In [11] Redmond explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. He gave an example of rings R and S and ideals $I \trianglelefteq R$ and $J \trianglelefteq S$, where $\Gamma(R/I) \cong \Gamma(S/J)$ but $\Gamma_I(R) \ncong \Gamma_J(S)$. Among other things, he showed that for an ideal I of R, $\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $\Gamma(R/I)$. In section 2, we show that for finite ideals I and J of R and S, respectively, for which $I = \sqrt{I}$ and $J = \sqrt{J}$, if $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$. Also we will show that the converse of this result holds if $|I| = |J|$ (see Theorem 2.2).

For a graph G, the vertices set of G is denoted by $V(G)$. The *degree* of a vertex v in G is the number of edges of G incident with v. We denote by $\delta(G)$ the minimum degree of vertices of G. For any nonnegative integer r, the graph G is called *r-regular* if the degree of each vertex is equal to r. The *girth* of G is the length of a shortest cycle in G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we define the girth of G to be infinite. An *r-partite* graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is jointed to every vertex that is not in the same subset. The *complete bipartite* (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is jointed by an edge is called a *complete* graph. We use K_n for the complete graph with n vertices. In section 3, we show that $\Gamma_I(R)$ is a complete bipartite graph provided $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \neq 0$ for prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R (see Theorem 3.1).

A *clique* of a graph is a maximal complete subgraph and the number of graph vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. In section 4, we show that if I is an ideal of R such that $I = \bigcap_{1\leq i\leq n} \mathfrak{p}_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1\leq i\leq n,\ i\neq j} \mathfrak{p}_i$ where $\mathfrak{p}_i$'s are prime ideals of R, then $\omega(\Gamma_I(R)) = n$ (see Theorem 4.2).

In this article the notations of graph theory are from [8], and the notations of commutative rings are from [9].

2 Some Basic Properties of Zero-Divisor Graphs

One of the main questions in the study of zero-divisor graphs is as follows: *Let R and S be two commutative rings. If $\Gamma(R) \cong \Gamma(S)$, then do we have $R \cong S$?* Some well known results on this question are as follows:

(i) If R and S are two finite reduced rings which are not fields, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see [4, Theorem 4.1]).

(ii) If R is a finite reduced ring which is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_6$, and S is a ring which is not a local integral domain, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see [2, Theorem 5]).

(iii) If $R = \prod_{i\in I} F_i$ and $S = \prod_{j\in J} G_j$, where F_i's are finite fields and G_j's are integral domains, and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see [5, Theorem 2.1]).

Now let I be an ideal of R and J be an ideal of S. It is natural to ask the following question. *If $\Gamma_I(R) \cong \Gamma_J(S)$, then do we have $R/I \cong S/J$?* The main purpose of this section is to focus on this question.

A subgraph H of G is called a *spanning* subgraph when $V(G) = V(H)$. A 1-regular spanning subgraph H of G is called a 1-*factor* or a *perfect matching* of G. A graph G is 1-*factorable* if the edges of G are partitioned into 1-factors of G. Every r-regular bipartite graph is 1-factorable (cf. [8, p. 192]). If the edges of G are partitioned into subgraphs $H_1, \ldots, H_n$, then we write $G \cong H_1 \oplus \ldots \oplus H_n$, and if $H_i \cong H_j$ for all $1 \leq i, j \leq n$, then we write $G \cong nH$, where $H \cong H_i$.

Theorem 2.1 *Let I be a finite ideal of R such that $I = \sqrt{I}$. Then $\Gamma_I(R) \cong |I|^2\Gamma(R/I)$.*

Proof. Let e be the edge of $\Gamma(R/I)$ between the vertices a and b. Since every element of coset $a + I$ is adjacent to every element of coset $b + I$, it is easy to see that there exists a subgraph of $\Gamma_I(R)$, denoted by $H^{(e)}$, which is isomorphic to complete bipartite graph $K_{|I|,|I|}$. On the other hand, by [8, p. 192], we have $K_{|I|,|I|} \cong {M_1}^{(e)} \oplus \ldots \oplus {M_{|I|}}^{(e)}$, where each of ${M_i}^{(e)}$ is a perfect matching of $K_{|I|,|I|}$. Now consider $K_i := \oplus_{e\in \mathrm{E}(\Gamma(R/I))} {M_i}^{(e)}$ which is a subgraph of $\Gamma_I(R)$. Since $I = \sqrt{I}$, $\Gamma_I(R) \cong K_1 \oplus \ldots \oplus K_{|I|}$. Now the assertion follows from the fact that each K_i is partitioned into $|I|$ edge-disjoint subgraphs, where each of them is isomorphic to $\Gamma(R/I)$. $\square$

Let S be a nonempty set of vertices of a graph G. The *subgraph induced by S* is the maximal subgraph of G with vertex set S, and is denoted by $\langle S\rangle$, that is, $\langle S\rangle$ contains precisely those edges of G joining two vertices in S.

Theorem 2.2 *Let I be a finite ideal of R and let J be a finite ideal of S such that $I = \sqrt{I}$ and $J = \sqrt{J}$. Then the following hold:*

(a) *If $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$.*

(b) *If $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$.*

Proof. Part (a) is an easy consequence of Theorem 2.1. For proving part (b), let $\varphi: \Gamma_I(R) \longrightarrow \Gamma_J(S)$ be an isomorphism. Now consider $K \subseteq R$ to be a set of distinct representatives of the vertices of $\Gamma(R/I)$. Clearly, the subgraph induced by K is isomorphic to $\Gamma(R/I)$. Now consider the restriction of φ to K. Suppose that $\varphi(K) = K'$ and $\langle K' \rangle = H$. Now, if $a, b \in V(K')$, then $a + J \neq b + J$; otherwise, $a^2 \in J = \sqrt{J}$, and hence $a \in J$, which is a contradiction. Hence, K' is a distinct representation of the vertices of $\Gamma(S/J)$, and hence $\langle K' \rangle = H \cong \Gamma_J(S)$. Therefore, φ induced an isomorphism from $\Gamma(R/I)$ to $\Gamma(S/J)$. $\square$

Note that in Theorem 2.2 (a), the condition "$|I| = |J|$" is not superficial, as the following example shows.

Example 2.3 Let $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $S = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and consider $I = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $J = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Hence, $\Gamma(R/I) \cong \Gamma(S/J)$. But by computing the number of edges in each graph we have $\Gamma_I(R) \ncong \Gamma_J(S)$.

The conditions "$I = \sqrt{I}$" and "$J = \sqrt{J}$" on ideals I and J are also necessary in Theorem 2.2 (see [11, Remark 2.3]).

Theorem 2.4 *Let I be a nonzero ideal of R and $a \in \Gamma_I(R)$, adjacent to every vertex of $\Gamma_I(R)$. Then $(I : a)$ is a maximal element of the set $\{(I : x) | \ x \in R\}$. Moreover, $(I : a)$ is a prime ideal.*

Proof. Let $V = V(\Gamma_I(R))$. Choose $0 \neq x \in I$. It is easy to see that $a \neq a + x \in \Gamma_I(R)$. Thus $a(a + x) \in I$ and hence $a^2 \in I$. Therefore, $V \cup I = (I : a)$, and so for any $x \in R$, $(I : x)$ is contained in $V \cup I = (I : a)$. Thus the first assertion holds.

Now, we prove that $(I : a)$ is a prime ideal. Let $xy \in (I : a)$ and $x, y \notin (I : a)$. Therefore, $xya \in I$. If $ya \notin I$, then $x \in (I : ya)$. We know that $(I : a) \subseteq (I : ya)$, and therefore, $(I : a) = (I : ya)$. Hence, $x \in (I : a)$, which is a contradiction. $\square$

Theorem 2.5 *Let I be an ideal of R and let S be a clique in $\Gamma_I(R)$ such that $x^2 = 0$ for all $x \in S$. Then $S \cup I$ is an ideal of R.*

Proof. Suppose that $x, y \in S \cup I$. Consider the following three cases.

Case 1: If $x, y \in I$, then $x - y \in S \cup I$.

Case 2: If $x, y \in S$ with $x - y \notin I$, then for all $c \in S$, $c(x - y) \in I$ and hence $S \cup \{x - y\}$ is a clique. Now, since S is a clique, $x - y \in S$.

Case 3: If $x \in I$ and $y \in S$, then $x - y \notin I$, and hence for any $c \in S$, $c(x - y) \in I$. Therefore, $x - y \in S$.

Now, let $x \in S \cup I$ and $r \in R$. Suppose that $r, x \notin I$. If $rx \in I$, then $rx \in S \cup I$.

If $rx \notin I$, since for any $c \in S$, $rxc \in I$, we have $rx \in S$. $\square$

Theorem 2.6 *Let I be an ideal of R and consider $S = \sqrt{I} \setminus I$. If S is a nonempty set, then $\langle S \rangle$ is connected.*

Proof. Let $x, y \in S$. If $xy \in I$, then the result is obtained. Suppose that $xy \notin I$, where $x^n, y^m \in I$ and $x^{n-1}, y^{m-1} \notin I$. Hence, the path

$$x — x^{n-1} — xy — y^{m-1} — y$$

is a path of length four from x to y. $\square$

Corollary 2.7 *Suppose either N is the nil radical of R, or is a nilpotent ideal of R. If N is nontrivial, then $\langle N \setminus \{0\} \rangle$ is a connected subgraph of $\Gamma(R)$.*

3 Complete r-Partite Graph

It is easy to see that if I is a prime ideal of R, then we have $\Gamma_I(R) = \emptyset$. In the following, we show that if $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals of R, then $\Gamma_I(R)$ is a complete bipartite graph. In section 4, we study the girth and the clique number of $\Gamma_I(R)$ for $I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$, where $\mathfrak{p}_i$'s are prime ideals of R.

Theorem 3.1 *Let I be a nonzero ideal of R. Then the following hold:*

(a) *If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals of R and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \neq 0$, then $\Gamma_I(R)$ is a complete bipartite graph.*

(b) *If $I \neq 0$ is an ideal of R for which $I = \sqrt{I}$, then $\Gamma_I(R)$ is a complete bipartite graph if and only if there exist prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R such that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$.*

Proof. (a): Let $a, b \in R \setminus I$ with $ab \in I$. Then $ab \in \mathfrak{p}_1$ and $ab \in \mathfrak{p}_2$. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, we have $a \in \mathfrak{p}_1$ or $b \in \mathfrak{p}_1$ and $a \in \mathfrak{p}_2$ or $b \in \mathfrak{p}_2$. Therefore, suppose $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ and $b \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Thus, $\Gamma_I(R)$ is a complete bipartite graph with parts $\mathfrak{p}_1 \setminus \mathfrak{p}_2$ and $\mathfrak{p}_2 \setminus \mathfrak{p}_1$.

(b): Suppose that the parts of $\Gamma_I(R)$ are V_1 and V_2. Set $\mathfrak{p}_1 = V_1 \cup I$ and $\mathfrak{p}_2 = V_2 \cup I$. It is clear that $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$. We now prove that $\mathfrak{p}_1$ is an ideal of R. To show this let $a, b \in \mathfrak{p}_1$.

Case 1: If $a, b \in I$, then $a - b \in I$ and so $a - b \in \mathfrak{p}_1$.

Case 2: If $a, b \in V_1$, then there is $c \in V_2$ such that $ca \in I$ and $cb \in I$. So, $c(a - b) \in I$. If $a - b \in I$, then $a - b \in \mathfrak{p}_1$. Otherwise, $a - b \in V_1$, which implies $a - b \in \mathfrak{p}_1$.

Case 3: If $a \in V_1$ and $b \in I$, then $a - b \notin I$, so there is $c \in V_2$ such that $c(a-b) \in I$. This implies that $a - b \in V_1$, and so $a - b \in \mathfrak{p}_1$.

Now let $r \in R$ and $a \in \mathfrak{p}_1$.

Case 1: If $a \in I$, then $ra \in I$ and so $ra \in \mathfrak{p}_1$.

Case 2: If $a \in V_1$, then there exists $c \in V_2$ such that $ca \in I$. So, $c(ra) \in I$. If $ra \in I$, then $ra \in \mathfrak{p}_1$ and if $ra \notin I$, then $ra \in V_1$ which implies $ra \in \mathfrak{p}_1$. Therefore, $\mathfrak{p}_1 \trianglelefteq R$.

We now prove $\mathfrak{p}_1$ is prime. For proving this let $ab \in \mathfrak{p}_1$ and $a, b \notin \mathfrak{p}_1$. Since $\mathfrak{p}_1 = V_1 \cup I$, $ab \in I$ or $ab \in V_1$, and so in any case there exists $c \in V_2$ such that $c(ab) \in I$. Thus $a(cb) \in I$. If $cb \in I$, then by the definition of $\Gamma_I(R)$ we have $b \in V_1$, that is a contradiction. Hence, $cb \notin I$ and so $cb \in V_1$. Therefore, $c^2 b \in I$. Since $I = \sqrt{I}$, $c^2 \notin I$. Hence, $c^2 \in V_2$. So $b \in V_1$ which is a contradiction. Therefore, $\mathfrak{p}_1$ is a prime ideal of R. $\square$

Note that if we consider $R = \mathbb{Z}_8$ and $I = \langle 4 \rangle$, then it is easy to see that $\Gamma_I(R)$ is bipartite, but I cannot be written as the intersection of two prime ideals. Therefore, the converse of Theorem 3.1 (a) is not valid in general. Hence, the condition "$I = \sqrt{I}$" on ideal I is not superficial in Theorem 3.1 (b).

Theorem 3.2 *Let I be a nonzero proper ideal of R. If $\Gamma_I(R)$ is a complete r-partite graph, $r \geq 3$, then at most one of the parts has more than one vertex.*

Proof. Assume that $V_1, \ldots, V_r$ are parts of $\Gamma_I(R)$. Let V_t and V_s have more than one element. Choose $x \in V_t$ and $y \in V_s$. Let V_l be a part of $\Gamma_I(R)$ such that $V_l \neq V_t$ and $V_l \neq V_s$. Let $z \in V_l$. Since $\Gamma_I(R)$ is a complete r-partite graph, $(I : x) = (\bigcup_{1 \leq i \leq r,\ i \neq t} V_i) \cup I$, $(I : y) = (\bigcup_{1 \leq i \leq r,\ i \neq s} V_i) \cup I$, and $(I : z) = (\bigcup_{1 \leq i \leq r,\ i \neq l} V_i) \cup I$. Therefore, $(I : z) \subseteq (I : x) \cup (I : y)$, and so we have $(I : z) \subseteq (I : x)$ or $(I : z) \subseteq (I : y)$. Let $(I : z) \subseteq (I : x)$ and choose $x' \in V_t$ such that $x' \neq x$. Then we have $x' \in (I : z) \setminus (I : x)$. This is a contradiction. $\square$

4 Girth and Clique Number

In this section we study the girth and the clique number of $\Gamma_I(R)$, when I is an intersection of prime ideals.

Theorem 4.1 *Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals of R and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Then either* $\mathrm{gr}(\Gamma_I(R)) = 4$ *or* $R/I \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. If $|\mathfrak{p}_1 \setminus \mathfrak{p}_2| = 1$ and $|\mathfrak{p}_2 \setminus \mathfrak{p}_1| \geq 2$, then $\Gamma_I(R)$ is a star graph and so has a cut point. This is a contradiction by [11, Theorem 3.2]. Therefore, this case cannot happen. The case $|\mathfrak{p}_2 \setminus \mathfrak{p}_1| = 1$ and $|\mathfrak{p}_1 \setminus \mathfrak{p}_2| \geq 2$ is similar. So there are two other possibilities.

Case 1: $|\mathfrak{p}_i \setminus \mathfrak{p}_j| \geq 2$ for $i \neq j$ and $1 \leq i, j \leq 2$. In this case, Theorem 3.1 implies that $\mathrm{gr}(\Gamma_I(R)) = 4$.

Case 2: $|\mathfrak{p}_i \setminus \mathfrak{p}_j| < 2$ for $i \neq j$ and $1 \leq i, j \leq 2$. In this case, there is $x \in R$ for which $\mathfrak{p}_1 \setminus \mathfrak{p}_2 = \{x\}$ and so $\mathfrak{p}_1 = \{x\} \cup I$. For any $r \in R \setminus \mathfrak{p}_2$ we have $rx \in \mathfrak{p}_1 \setminus I$ and so $rx = x$. Therefore, $(1 - r)x = 0 \in \mathfrak{p}_2$ and hence $(1 - r) \in \mathfrak{p}_2$. Thus $|R/\mathfrak{p}_2| = 2$. That implies $\mathfrak{p}_2$ is a maximal ideal of R and $R/\mathfrak{p}_2 \cong \mathbb{Z}_2$. But $\mathfrak{p}_1 + \mathfrak{p}_2 = R$, so that implies $R/I \cong R/\mathfrak{p}_1 \times R/\mathfrak{p}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\square$

Theorem 4.2 *Let I be an ideal of R such that $I = \bigcap_{1 \leq i \leq n} \mathfrak{p}_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n,\ i \neq j} \mathfrak{p}_i$ where $\mathfrak{p}_i$'s are prime ideals of R. Then $\omega(\Gamma_I(R)) = n$.*

Proof. Consider $x_j \in \bigcap_{1 \leq i \leq n,\ i \neq j} \mathfrak{p}_i \setminus \mathfrak{p}_j$. It is easy to see that $X = \{x_1, \ldots, x_n\}$ is a clique in $\Gamma_I(R)$. Hence, $\omega(\Gamma_I(R)) \geq n$ and so it is sufficient to show that $\omega(\Gamma_I(R)) \leq n$. In order to do this, we use induction on n. For $n = 2$, by Theorem 3.1, $\Gamma_I(R)$ is a bipartite graph and hence $\omega(\Gamma_I(R)) = 2$. Suppose $n > 2$ and the result is true for any integer less than n. Let $I = \bigcap_{1 \leq i \leq n} \mathfrak{p}_i$ and for each $1 \leq j \leq n$, $I \neq \bigcap_{1 \leq i \leq n,\ i \neq j} \mathfrak{p}_i$. Let $\{x_1, \ldots, x_m\}$ be a clique in $\Gamma_I(R)$. Hence, $x_1 x_j \in \bigcap_{1 \leq i \leq n} \mathfrak{p}_i$ for any $2 \leq j \leq m$. Without loss of generality, suppose that $x_1 \notin \mathfrak{p}_1$. Therefore, $x_2, \ldots, x_m \in \mathfrak{p}_1$, so $x_2, \ldots, x_m \notin \bigcap_{2 \leq i \leq n} \mathfrak{p}_i$. Let $J = \bigcap_{2 \leq i \leq n} \mathfrak{p}_i$. Hence, $\{x_2 \ldots, x_m\}$ is a clique in $\Gamma_J(R)$. Therefore, $m - 1 \leq n - 1$, and the result is obtained. $\square$

Corollary 4.3 *The following hold:*

(a) *If $I = \bigcap_{1 \leq i \leq n} \mathfrak{p}_i \neq 0$ and $J = \bigcap_{1 \leq j \leq m} \mathfrak{q}_j$ where $\mathfrak{p}_i$'s and $\mathfrak{q}_j$'s are prime ideals such that $\Gamma_I(R) = \Gamma_J(R)$, then $m = n$.*

(b) *If for any $\mathfrak{p} \in \mathrm{Min}(R)$, $\mathfrak{p}$ is a finitely generated ideal, then $\omega(\Gamma_{\mathrm{nil}(R)}(R)) = |\mathrm{Min}(R)|$ (which is finite by the main theorem of* [3]*).*

(c) *If R is a semi-local ring and not local, then $\omega(\Gamma_{J(R)}(R)) = |\mathrm{Max}(R)|$.*

(d) *If n is a square-free integer, then $\omega(\Gamma_{n\mathbb{Z}}(\mathbb{Z})) = k$, where k is the number of primes in the decomposition of n into primes.*

Theorem 4.4 *Let I be an ideal of R. Suppose either I is a primary ideal of R that is not prime and $|I| \geq 3$, or $|\mathrm{Ass}(R/I)| \geq 3$. Then $\mathrm{gr}(\Gamma_I(R)) = 3$.*

Proof. For the first case, let $a, b \in R \setminus I$ such that $ab \in I$. Then there exists

$n \in \mathbb{N}$ such that $b^n \in I$, so we can choose $t \in \mathbb{N}$ for which $b^t \in I$ and $b^{t-1} \notin I$. Since $a, b^{t-1} \notin I$, we have the chain

$$a — b — b^{t-1} — a$$

in the graph $\Gamma_I(R)$. Therefore, $\mathrm{gr}(\Gamma_I(R)) = 3$.

For the second case, $|\mathrm{Ass}(R/I)| \geq 3$ implies that $\mathrm{gr}(\Gamma(R/I)) = 3$ (see [1, Corollary 2.2]), and hence $\mathrm{gr}(\Gamma_I(R)) = 3$. $\square$

In the above theorem, one of the conditions "$|I| \geq 3$" or "$|\mathrm{Ass}(R/I)| \geq 3$" are necessary. To see this, for example let $R = \mathbb{Z}_8$ and consider $I = \langle 4 \rangle$; and note that we have $|\mathrm{Ass}(R/I)| = 1$ and $\mathrm{gr}(\Gamma(R/I)) = \infty$.

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