ZERO ENTROPY FACTORS OF TOPOLOGICAL FLOWS

F.Blanchard & Y. Lacroix[(⋆)]

Abstract. The maximal zero entropy factor of a topological flow is defined using entropy pairs, and explicitly given for some cartesian products. As a consequence, it is proved that only the trivial flow is disjoint from all flows whose maximal zero entropy factor is trivial.

0. Introduction.

A topological flow is a pair (X,T) where X is a compact metric space and T is a homeomorphism of X to itself. The topological entropy of the flow (X,T) is defined in [Ad-Ko-McAn]. It shall be denoted by $h(X,T)$ (if (U,V) is a cover of X, denote by $h(U,V)$ the entropy of (U,V)).

Recall (Y,S) is a factor of (X,T) if there exists a continuous, onto map $\phi : (X,T) \longrightarrow (Y,S)$ satisfying the commutation relation $\phi \circ T = S \circ \phi$.

In the metric theory of dynamical systems, the maximal zero entropy factor is well known; it is determined by the Pinsker $\sigma-$algebra. However, in Topological Dynamics, the corresponding object has not yet been proved to exist.

The central point of this paper is the definition of the maximal zero entropy factor of a topological flow. Its construction is based on the use of entropy pairs, introduced in [Bla-2]; it is the factor associated to the smallest closed equivalence relation collapsing all entropy pairs. This is to be compared to the way Ellis and Gottschalk proved the existence of a maximal equicontinuous factor in flows ([El-Go]).

The paper is organized as follows. In Part I, we give preliminary definitions concerning entropy pairs, u.p.e. and c.p.e. flows. In Part II, we examine the relation between topological factors of a flow and closed invariant equivalence relations in the product. This and entropy pairs make it possible to define the maximal zero entropy factor of a flow. In Part III, we give some applications, more or less close to the notion of maximal zero entropy factor. The first is Prop. 3.1.: the maximal zero entropy factor of the product of a flow (X,T) with u.p.e. and a flow (Y,S) with zero entropy and a dense union of minimal subsets is (Y,S) (second coordinate projection). The second is an example illustrating the necessity of assuming the union of minimal subsets is dense in Prop. 3.1.. The third is an application of Prop. 3.1. to an example developped in [Bla-1], and gives an example of a flow for which the maximal zero entropy factor is strictly contained, as a factor, in the Pinsker σ-algebra. The fourth shows that the trivial flow is the only one disjoint from all flows having trivial maximal zero entropy factor. The Appendix gives a definition for uniform positive relative entropy, and that of "Entropy-flows".

The two properties of u.p.e. and c.p.e. were introduced in [Bla-1] as an attempt to give a topological counterpart to the K property of measurable dynamics: one can think of the maximal

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zero entropy factor as corresponding to the Pinsker $\sigma-$algebra; however, as shownby the example at the end of application 3, these two factors do not coincide.

C.p.e. is a very weak property (as shown by Proposition 3.3, or the facts about c.p.e. in [Bla-1]), which is not the case of the K property, though both may be formally defined in the same way; u.p.e., a perfectly natural definition in view of the properties of entropy pairs, has no obvious conterpart in the metric theory.

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I. Preliminaries.

Most of the following definitions and statements are taken from [Bla-2];

– a standard cover of X is a pair (U, V) of two non dense open sets of X, covering X.

– if $x, x' \in X$, and (U, V) is a standard cover of X, we shall say that x and x' are distinguished by the cover (U, V) if and only if $x \in \mathrm{Int}(U^c)$ and $x' \in \mathrm{Int}(V^c)$.

– a pair of points (x, x') in X^2 is called an entropy pair if and only if any standard cover distinguishing x and x' has positive entropy.

Let us define the following set;

$$E(X,T) = \{(x,x') \in X^2, (x,x') \text{ is an entropy pair}\}.$$

Denoting by ˙ the diagonal subset of X^2, the set $E(X,T) \cup \Delta$ is a closed $T \times T-$invariant subset of X^2.

A flow (X, T) has uniform positive entropy (abbreviate "has u.p.e.") if and only if any standard cover of (X, T) has strictly positive entropy (in [Bla-1], this property was shown to imply weak mixing).

One easily checks that (X, T) has u.p.e. if and only if $E(X,T) \cup \Delta = X^2$. One proves that $h(X,T) > 0$ if and only if $E(X,T) \neq \emptyset$.

A flow (X, T) is said to have c.p.e. (Completely Positive Entropy) if and only if all its non trivial factors have positive entropy.

If $\phi : (X,T) \to (Y,S)$ is a factor, then the two following properties hold;

$$\begin{cases} \textit{Property } (a): \ \textit{if } (x,x') \in E(X,T), \textit{ and } \varphi(x) \neq \varphi(x'), \textit{ then } (\varphi(x), \varphi(x')) \in E(Y,S), \\ \\ \textit{Property } (b): \ \textit{if } (y,y') \in E(Y,S), \textit{ then } \exists x, x' \in X, \ \varphi(x) = y, \ \varphi(x') = y', \\ \qquad\qquad\qquad \textit{and } (x,x') \in E(X,T). \end{cases}$$

Here is another elementary property of entropy pairs, which proves useful in the sequel ([B:a-2, Prop. 5]);

Proposition 1.1. *Suppose X' is a closed $T-$invariant subset of (X,T). Then if (x,x') is an entropy pair of (X',T), it is also an entropy pair of (X,T).*

II. Closed and equivalence $T \times T$–invariant relations.

Let (X, T) be a flow and $G \subset X \times X$ a closed $T \times T$–invariant equivalence relation. Denote the associated quotient space $X/_G$ and the associated quotient map $p_G : X \to X/_G$.

Assume metric $d(.,.)$ defines the given compact topology on X. If $x \in X$, let $\bar{x}$ be the closed set $\{y \in X, \ (x,y) \in G\}$; then define, for $\bar{x}, \ \bar{y} \in X/_G$,

$$\bar{d}(\bar{x}, \bar{y}) := d(\bar{x}, \bar{y}).$$

This defines a distance $\bar{d}$ on $X/_G$; the compact topology it induces on this set providing is the usual quotient topology; furthermore, the map p_G is continuous, and T induces a homeomorphism on $X/_G$, denoted $\bar{T}$, and defined unambiguously in the following way;

$$\bar{T}(\bar{x}) := p_G(Tx),$$

for any $\bar{x} \in X/_G$. These constructions lead to the following commutative diagram of continuous onto maps;

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle p_G}\big\downarrow & & \big\downarrow{\scriptstyle p_G} \\ X/_G & \xrightarrow{T/_G} & X/_G \end{array}$$

Conversely, any factor $\phi : (X,T) \to (Y,S)$ determines a closed $T \times T$–invariant equivalence relation $G_\phi \subset X \times X$:

$$(\ \phi(x) = \phi(x')\ ) \Leftrightarrow (\ (x,x') \in G_\phi\ ),$$

and $(X/_{G_\phi}, T/_{G_\phi})$ is topologically conjugate to (Y,S) via the map ϕ';

$$\phi'(p_{G_\phi}(x)) := \phi(x),\ x \in X.$$

Thus, the factors of (X,T) are in a one-to-one correspondance with the closed $T \times T$–invariant equivalence relations in $X \times X$.

Assume G and G' are two such, and that furthermore $G' \subset G$; then one can define the map $p_{G,G'} : X/_{G'} \to X/_G$ by the formula

$$p_{G,G'}(p_{G'}(x)) := p_G(x),\ x \in X.$$

Then we have the following commutative diagram;

$$\begin{array}{ccc} (X,T) & \xrightarrow{Id_X} & (X,T) \\ {\scriptstyle p_{G'}}\big\downarrow & & \big\downarrow{\scriptstyle p_G} \\ (X/_{G'}, T/_{G'}) & \xrightarrow{p_{G,G'}} & (X/_G, T/_G) \end{array}$$

Reciprocally, if

$$\begin{array}{ccc} (X,T) & \xrightarrow{Id_X} & (X,T) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ (Z,R) & \xrightarrow{g} & (Y,S) \end{array}$$

is a commutative diagram of factors, then $G_f \subset G_\phi$.

Next, if R is a subset of $X \times X$, and $\mathcal{F}(R)$ is the set of closed $T \times T$–invariant equivalence relations containing R, then $\mathcal{F}(R)$ is not empty. Moreover,

$$G(R) := \left(\bigcap_{G \in \mathcal{F}(R)} G \right) \in \mathcal{F}(R),$$

is the smallest element of $\mathcal{F}(R)$ partially ordered with inclusion. Let

$$G(R) := \bigcap_{G \in \mathcal{F}(R)} G.$$

Define

$$G_0(X,T) = G\big(E(X,T)\big).$$

For a flow (X,T), let $(X_0, T_0) := (X/_{G_0(X,T)}, T/_{G_0(X,T)})$, and $p_0 = p_{G_0(X,T)}$.

Theorem 2.1. *Let $\phi : (X,T) \to (Y,S)$ be a factor map. Then $h(Y,S) = 0$ if and only if there exists a factor map $f : (X_0,T_0) \to (Y,S)$ such that the following diagram is commutative;*

$$\begin{array}{ccc} (X,T) & \xrightarrow{Id_X} & (X,T) \\ {\scriptstyle p_0}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ (X_0,T_0) & \xrightarrow{f} & (Y,S) \end{array}$$

Proof. Assume $h(Y,S) = 0$. Then by definition of entropy pairs $E(Y,S) = \emptyset$. By Property (a) this means that if $(x,x') \in E(X,T)$ one must have $\phi(x) = \phi(x')$: hence $G_0(X,T) \subset G_\phi$, and from the remarks above there follows existence of a map f with the suitable properties.

Conversely, suppose (Y,S) is a factor of (X_0,T_0). One has $h(X_0,T_0) = 0$: indeed, if this were false, $E(X_0,T_0)$ would contain some pair (z,z') with $z \neq z'$; by Property (b) this means some entropy pair of (X,T) would have image (z,z') by p_0, thus not being collapsed, which contradicts the definition of p_0. Therefore $h(Y,S)$ is also 0. □

Definition 2.1. *(X_0,T_0) is called the maximal zero entropy factor of (X,T).*

Remark 2.1. *M. Lemanczyk suggested another proof of Theorem 2.1., using the variationnal principle.*

III. Applications.

It is now clear that the determination of the maximal zero entropy factor of a flow relies upon the localisation of its entropy pairs.

Application 1.

Let (X,T) have u.p.e. and $h(Y,S) = 0$. One would expect the maximal zero entropy factor of the product flow $(X \times Y, T \times S)$ to be (Y,S). Surprisingly, this is not always the case, and requires an additional condition. Recall that a minimal subset of a flow is one with no proper closed invariant subset;

Proposition 3.1. *Let (X,T) have u.p.e., $h(Y,S) = 0$ and assume the union of minimal subsets of (Y,S) is dense. Then*

$$E(X \times Y, T \times S) \cup \Delta_{X\times Y} = \left\{ \big((x,y),(x',y)\big);\ x,x' \in X,\ y \in Y \right\},$$

and the maximal zero entropy factor of $(X \times Y, T \times S)$ is (Y,S).

Proof. First suppose $h(Y,S) = 0$ only. The projections of $(X \times Y, T \times S)$ on (X,T) and (Y,S) are factors. Suppose $\big((x,y),(x',y')\big) \in E(X \times Y, T \times S) \cup \Delta$; then by Property (a), either $(y,y') \in E(Y,S)$ or $y = y'$. Since $h(Y,S) = 0$, $E(Y,S) = \emptyset$, and we obtain $E(X \times Y, T \times S) \cup \Delta_{X\times Y} \subset \left\{ \big((x,y),(x',y)\big);\ x,x' \in X,\ y \in Y \right\}$.

In order to prove the reverse inclusion, let us first use the u.p.e. assumption on (X,T). As u.p.e. implies weak mixing ([Bla-1], Proposition 2) there is some pair $(x,x') \in X^2$ having dense orbit under $T \times T$. It cannot belong to the diagonal, so $x \neq x'$ and (x,x') is an entropy pair of (X,T). Thus, by Property (b), there is $y \in Y$ such that $\big((x,y),(x',y)\big) \in E(X \times Y, T \times S)$. Now, $T \times Id_Y$ is an automorphism of the product flow; by Property (a), all points in the orbit of $\big((x,y),(x',y)\big)$ under $T \times Id_Y$ belong to $E(X \times Y, T \times S) \cup \Delta_{X\times Y}$. On account of the choice of (x,x'), its closure under action of $T \times Id_Y$ is all of $(X \times \{y\}, X \times \{y\})$, so $(X \times \{y\}, X \times \{y\}) \subset E(X \times Y, T \times S) \cup \Delta_{X\times Y}$.

There remains to prove this is true for any $y \in Y$: this is where the additional assumption on (Y,S) comes in. When (Y,S) is minimal it is sufficient to apply Property (a) to the action of the endomorphism $Id_X \times S$ on $(X \times \{y\}, X \times \{y\})$, and then closedness of $E(X \times Y, T \times S) \cup \Delta_{X \times Y}$. If (Y,S) merely contains a dense union of minimal (Y_i, S), $i \in I$, we may apply the last result to each $(X \times Y_i, T \times S)$, $i \in I$, and then Proposition 1.1.

$E(X \times Y, T \times S) \cup \Delta_{X \times Y}$ is obviously the graph of an equivalence relation. Call ϕ the associated homomorphism; ϕ identifies points having the same projection on Y, and never identifies points having distinct projections on Y. This means the quotient space is (Y,S). □

When the union of minimals subsets is not dense in Y, the conclusion of Proposition 3.1. does not hold. The following example illustrates this claim.

Application 2.

Let (X,T) have u.p.e., and $Y = \mathbb{N} \cup \{\infty\}$ be the Alexandroff compactification of $\mathbb{N}$. Let S be the translation by 1 on Y, having ∞ as unique fixed point. (Y,S) is not transitive, and $h(Y,S) = 0$ as will be seen below. The closed invariant set $X \times \{\infty\}$ is conjugate to X inside $(X \times Y, T \times S)$, so its cartesian square is contained in $E(X \times Y, T \times S) \cup \Delta$ by Proposition 1.1.

Suppose (U,V) is any standard cover of X; if $i \in \mathbb{N}$, $U^c \times \{i\}$ and $V^c \times \{i\}$ are two closed sets of $X \times Y$ with non empty interior, so that $U' := (U^c \times \{i\})^c$ and $V' := (V^c \times \{i\})^c$ form a standard cover $\mathcal{R}$ of $X \times Y$. This cover has entropy zero, merely on account of the second coordinate; for $x \in X$, $n \in \mathbb{N}$, $(x,n) \in (T \times S)^{-k}(U' \cap V')$ whenever $n + k \neq i$; if $k = i - n$, then $(x,n) \in (T \times S)^{n-i}(U')$ or $(x,n) \in (T \times S)^{n-i}(V')$ according to the position of x in X. Now choose some p and consider cover $\mathcal{R}_p$. All points whose orbit from 0 to $p-1$ never reaches $V^c \times \{i\}$, $i = o, \ldots, p-1$ are in the set $U' \cap (T \times S)^{-1}(U') \cap \ldots \cap (T \times S)^{-p+1}(U')$; the others belong to one of the sets $U' \cap \ldots \cap (T \times S)^{-i+1}(U') \cap (T \times S)^{-i}(V') \cap (T \times S)^{-i-1}(U') \cap \ldots \cap (T \times S)^{-p+1}(U')$. We have thus found a subcover of $\mathcal{R}_p$ with cardinality $p+1$, and $h(\mathcal{R}, T \times S) = 0$.

For $x \neq x'$, $\mathcal{R}$ can be chosen so as to distinguish (x,i) and (x',i), which implies there is no entropy pair of the form $\big((x,i),(x',i)\big)$, $i \in \mathbb{N}$, $x \neq x'$. As was remarked in the proof of Proposition 3.1 no pair of the form $\big((x,i),(x,i')\big)$, $i \neq i'$ may be in $E(X \times Y, T \times S)$. Therefore, $E(X \times Y, T \times S) \cup \Delta = \big(X \times \{\infty\}\big)^2$.

$\big(X \times \{\infty\}\big)^2 \cup \Delta$ is the graph of an equivalence relation. The associated homomorphism ϕ identifies all points (x,∞), and them only. In the flow $(X \times Y, T \times S)$ the maximal zero entropy factor is therefore much bigger than (Y,S). □

Application 3.

Here is an application of Proposition 3.1.

Let Y be a subshift of $A^{\mathbb{Z}}$, $0 \notin A$, and let $\phi : \{0,1\}^{\mathbb{Z}} \times Y \longrightarrow (A \cup \{0\})^{\mathbb{Z}}$ be defined by the alphabetic map $\overline{\phi} : \{0,1\} \times A \to A \cup \{0\}$: $\overline{\phi}(1,a) = a$, $\overline{\phi}(0,a) = 0$, $a \in A$. Let $Z = \phi(\{0,1\}^{\mathbb{Z}} \times Y)$.

It was shown in [Bla-1, Proposition 10] that Z has c.p.e. (i.e. its maximal zero entropy factor is trivial) whenever Y is minimal, but also that it has not u.p.e. when $h(Y) = 0$. This is obvious when one looks at entropy pairs. If (Y,σ) is minimal with entropy 0, since $\{0,1\}^{\mathbb{Z}}$ has u.p.e., by Proposition 3.1., $E(X \times Y, \sigma \times \sigma) \cup \Delta = \big\{ \big((x,y),(x',y)\big);\ x,x' \in X,\ y \in Y \big\}$. Call z_0 the fixed point on letter 0 in Z, x_0, x_1 the fixed points on letters 0 and 1 in $\{0,1\}^{\mathbb{Z}}$; given $z \in Z$ one easily constructs a unique point $x \in \{0,1\}^{\mathbb{Z}}$ such that $z = \phi(x,y)$ by replacing all symbols of A in z by 1's, but y is not generally unique; nevertheless Z contains a copy of Y: $\phi(x_1,y) = y$, and (x_1,y) is the only point with image y under ϕ. As $\big((x_1,y),(x_1,y')\big)$ never belongs to $E(X \times Y, \sigma \times \sigma)$ this implies $(y,y') \notin E(Z)$ for $y \neq y'$: therefore Z has not got u.p.e..

Nevertheless, its maximal zero entropy factor is trivial. For any $z = \phi(x,y) \neq z_0$, one has $((x_0,y),(x,y)) \in E(X \times Y, \sigma \times \sigma)$ by Proposition 3.1., hence, by Property (a), $(z_0,z) \in E(Z,\sigma)$.

The smallest equivalence relation containing $E(Z, \sigma)$ identifies all points with z_0, the associated homomorphism thus having trivial image.

And here is an application of the preceding one, giving an example of a flow for which the maximal zero entropy factor and the metric factor corresponding to the Pinsker σ-algebra of the flow do not coincide. This answers a question by M. Lemańczyk.

Let (Z', σ) be the symbolic flow on alphabet $\{0, 1\}$ associated to the following diagram;

The flow (Z', σ) is in fact the subshift containing all bi-infinite sequences $x = (x_n)_{n\in\mathbb{Z}}$ on $\{0, 1\}$ for which given any $p \in \mathbb{Z}$, and $n \in I\!\!N$, one can follow the arrows in figure 1, starting from state α or β, and read along the arrows the word $x_p, x_{p+1}, \cdots, x_{p+n}$.

It is obtained from the above construction by assuming $Y = \{p, q\}$, where p is the sequence having a's in odd positions and b's in even positions, and q is the shifted image of p; then construct the corresponding Z with map ϕ sending a to 1, b to 0, and 0 to 0.

This flow has only one fixed point $1^{\mathbb{Z}}$. Let

$$U = \{x \in \{0, 1\}^{\mathbb{Z}},\ x_n = 0 \Rightarrow n \text{ is odd},\ n \in \mathbb{Z}\},$$

and

$$V = \{x \in \{0, 1\}^{\mathbb{Z}},\ x_n = 0 \Rightarrow n \text{ is even},\ n \in \mathbb{Z}\}.$$

Then $\big(\{1^{\mathbb{Z}}\} \times U \cup V\big) \subset E(Z', \sigma)$, thus $G_0(Z', \sigma)$ must be equal to $Z' \times Z'$. Hence, (Z', σ) has trivial maximal zero entropy factor.

On the over hand, let $\mathcal{B}_{Z'}$ be the Borel σ-algebra of Z', and let μ be any ergodic σ-invariant probability borelian measure on Z', such that measure theoretically, (Z', σ, μ) is not trivial. Then $\mu(\{1^{\mathbb{Z}}\}) = 0$, and since $\sigma(U) = V$, $\sigma(V) = U$, the flow (Z', σ, μ) has, as a measure theoretic factor, the one with two points sending one on each other, each point with measure $\frac{1}{2}$. Therefore, the factor associated to the Pinsker σ-algebra of (Z', σ) contains this last flow, and is not trivial.

Application 4.

The proof of the next statement combines ideas from the proof of Proposition 3.1., and a construction generalising that of Example 3.

Two flows (X, T) and (Y, S) are said to be disjoint if there exists no proper closed $T \times S-$invariant subset of $(X \times Y, T \times S)$ having projections X and Y on the two coordinates. For two flows to be disjoint, one of them at least must be minimal ([Fur]).

Proposition 4.3. *The set of flows disjoint from all c.p.e. flows is reduced to the trivial flow.*

Proof. Flows disjoint from all c.p.e. flows are necessarily minimal, because some c.p.e. flows are not (the full shift for example). It is therefore sufficient to prove that for any minimal flow, one can construct a non disjoint c.p.e. flow.

Let (X, T) have u.p.e. and contain a fixed point t (take for instance a full shift) and (Y, S) be non trivial and minimal. Define the continuous function f from X to $[0, 1]$: $f(x) = d(t, x)$;

$\min f(x) = f(t) = 0$, and without loss of generality, one can assume $\max_x d(t,x) = 1$. Choose some point $u \in Y$ and a, $0 < a < 1$, and put $g(y) = a + d(u,y)$; g is continuous; one has $\min_y g(y) = g(u) = a$, $\max_y g(y) = b > a$ since Y is not a singleton. For a doubly infinite sequence ω, denote by ω_n its coordinate at time n. Functions f and g allow us to define a homomorphism ϕ from $(X \times Y, T \times S)$ onto some closed shift-invariant subset Z of $[0,b]^{\mathbb{Z}}$, endowed with the shift σ, by the formula $(\phi(x,y))_n = f(T^n x) g(S^n y)$, $n \in \mathbb{Z}$. The fixed image $s = \phi(t,y)$ for any $y \in Y$ is the fixed point on real number 0.

Let us prove Z has c.p.e.. For this we show that whatever $z \in Z$, $z \neq s$, $(s,z) \in E(Z,\sigma)$. Since $z \in Z$ there is a pair (x',y') such that $z = \phi(x',y')$, so $(s,z) = (\phi(t,y), \phi(x',y'))$ for any y and for some $x' \neq t$. As (X,T) has u.p.e., one has $(t,x') \in E(X,T)$, therefore by Property (b) there are $y_1,\ y_1' \in Y$ such that $((t,y_1),(x',y_1')) \in E(X \times Y, T \times S)$. Now since Y is minimal the orbit of y_1' under S is dense in Y; $Id \times S$ is an homomorphism of $(X \times Y, T \times S)$, so by Property (a) and closedness of $E(X \times Y, T \times S) \cup \Delta$ there is $y_2 \in Y$ such that $((t,y_2),(x',y')) \in E(X \times Y, T \times S)$; now this is enough: by Property (a) again $\big(\phi(t,y_2), \phi(x',y')\big) = (s,z) \in E(Z,\sigma)$. To finish this part of the proof remark that since all points $z \neq s$ in Z form an entropy pair with s, the smallest equivalence relation saturating $E(Z,\sigma)$ is trivial; so is the maximal zero entropy factor of (Z,σ).

Finally (Z,σ) is not disjoint from (Y,S): the set

$$F = \{\ (z,y) \in Z \times Y;\ \exists x \in X :\ \varphi(x,y) = z\ \} \subset Z \times Y$$

is obviously closed and invariant by $\sigma \times S$; it has image Z under projection on the first coordinate, since the defining condition rests on y alone, and image Y under projection on the second because $\phi(X \times Y) = Z$. F is also a proper subset of $Z \times Y$: there exists $z \in Z$ such that $z_0 = b > a$; then for any $x \in X$, $f(x).g(u) \leq a < b$, so that $(z,u) \notin F$. $\square$

Appendix.

The notion of relative $K-$systems has proved useful in Ergodic Theory; so might that of relative u.p.e. for flows. Here are two possible definitions.

Let $\phi : (X,T) \to (Y,S)$ be a factor; define the following relation;

$$(\ x\mathcal{R}_\phi x'\ ) \Leftrightarrow (\ (\ \phi(x) = \phi(x')\ )\ or\ (\ \phi(x) \neq \phi(x')\ and\ (\phi(x),\phi(x')) \in E(Y,S)\ )\ ).$$

Definition A.1. *We shall say that $\phi : (X,T) \to (Y,S)$ is a factor of uniform positive relative entropy (and abbreviate "ϕ is an f.u.p.r.e.") if the following holds;*

$$(\ (\ x\mathcal{R}_\phi x'\ )\ and\ (\ x \neq x'\ )\ ) \Leftrightarrow (\ (x,x') \in E(X,T)\ ).$$

In other words, the factor $\phi : (X,T) \to (Y,S)$ is an f.u.p.r.e. of (X,T) if and only if

$$\mathcal{R}_\phi \setminus \Delta = E(X,T).$$

With this first definition, we have the following;

Proposition A.1. *Let $\phi : (X,T) \to (Y,S)$ be an f.u.p.r.e.. Then p_0 factors through ϕ.*

Proof. It is enough to show that when ϕ is an f.u.p.r.e., $G_\phi \subset E(X,T) \cup \Delta$. By definition of an f.u.p.r.e. this condition is fulfilled. $\square$

One can prove the following, examinating all possible cases;

Proposition A.2. *Assume the following diagram of topological factors is commutative;*

$$\begin{array}{ccc} (X,T) & \xrightarrow{Id_X} & (X,T) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\ (Z,R) & \xrightarrow{g} & (Y,S) \end{array}$$

Then ϕ is an f.u.p.r.e. if and only if f and g both are.

Definition A.2. *A flow (X,T) shall be called an entropy flow, say "E-flow", if and only if $E(X,T)\cup\Delta$ is an equivalence relation.*

Property A.1. *An E-flow is an f.u.p.r.e. with respect to its maximal zero entropy factor.*

Proposition 3.1. provides some (rather naive) examples of f.u.p.r.e. and entropy flows.

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Université de Provence, Laboratoire de Mathématiques discrètes, CNRS, Case X, 3 place V.Hugo, 13331 Marseille Cedex 3, France.

U.B.O, Fac. des Sciences, Départ^t de Maths, 6 avenue V. Le Gorgeu, B.P. 809, 29285 Brest Cedex.