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Zero Mass Divergences in Expansions about Chiral and Scale Symmetry

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Abstract

Singularities of expansions in the symmetry breaking parameters of chiral and scale asymmetric theories are studied, in tree approximation, in polynomial Lagrangian models. The singularities are related to the appearance of zero mass scalar fields, not necessarily Goldstone particles, at the radius of convergence of the expansion. Methods of avoiding these singularities are presented.

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DISCLAIMER

I. Introduction

Analyticity of Nambu-Goldstone symmetry realizations[1] and the possibility of expansions[2] in symmetry breaking parameters have recently been studied in the context of pion loop diagrams[3] and the solution, in tree approximation, of specific polynomial Lagrangian models[4-7] built on $(3,3^*)\oplus(3^*,3)$ fields.[8] In the latter approach, with which we are concerned here, linear but spontaneously broken realizations of chiral and scale symmetry result in partial conservation conditions with pseudo-scalar masses determined quite simply by the symmetry nature of the ground state and the form of the symmetry breaking part of the Lagrangian[9-11] whereas the scalar masses, as well as details of the spontaneously broken solution are highly model dependent. Such models have a wide variety of solutions but are limited by such general considerations as those of Okubo and Mathur,[12] or, apparently equivalently in the tree approximation, by positivity requirements[5,11] on the square of the masses. An actual fit to all meson masses[4-7,11,13] further limits the Lagrangian and its spontaneously broken solutions by constraining the coefficients of the various symmetric and symmetry breaking terms. Thus if one arranges the number of such terms to be small enough, all coefficients may be determined, resulting in a complete theory whose analytic properties may be studied in various limits. It has been suggested that the properties found may be more general than the specific models from which they come. Here we wish to point out some general notions regarding analyticity in the symmetry breaking parameter of the one point function (and hence, usually, of the n-point functions) which guide one in constructing Lagrangians and imposing analytic structure, and apply these notions to current models of chiral and scale symmetry breaking. In general, we find that a singularity appears when a scalar mass vanishes, associated perhaps with an infrared divergence. (Consequently, the Nambu-Goldstone mechanism as applied to scale symmetry usually fails to have a power series expansion in the scale breaking parameter.) Exceptional cases exist, however, which allow a smooth limit when, by careful planning, the singularity is canceled by a zero. In particular, we wish to discuss analytic properties of the case in which $\langle o|u_8|o\rangle \equiv 0$ where bilinear (non-pole) terms are required[9,11] in the symmetry breaking part of the Lagrangian, allowing $(1,8)\oplus(8,1)$ contributions,[9,14] and non-unique c-parameters.

The variety of symmetry breaking structure and analytic behaviors possible in these models suggests that while it may be very difficult to abstract from them, a priori, detailed predictions on the possibility of expansions in the symmetry breaking parameters or even predictions of the symmetry breaking form of the non-pole terms, they will remain useful probes of chiral breakdown provided external input beyond scalar masses and PCAC conditions are imposed.

II. Connection between Singularities and Zeros of the Mass Matrix

Let us consider a Lagrangian $\mathcal{L}(\{\varphi\},\delta)$ which is a polynomial in a set of scalar and pseudoscalar fields $\varphi_i, i = 1,2\ldots$ and linear in a symmetry breaking parameter δ. Then

$$\frac{d}{d\delta}\frac{\partial\mathcal{L}}{\partial\varphi_i} = \frac{\partial}{\partial\varphi_j}\frac{\partial\mathcal{L}}{\partial\varphi_i}\frac{d\varphi_j}{d\delta} + \frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_i} \tag{1}$$

For fields $\tilde{\varphi}_i(\delta)$ which are solutions of the extremum equations

$$\tilde{\varphi}_i(\delta) \quad : \quad \frac{\partial \mathcal{L}}{\partial \varphi_i}(\{\varphi\},\delta) = 0 \tag{2}$$

we have $\frac{d}{d\delta}\frac{\partial \mathcal{L}}{\partial \varphi_i} = 0$ and thus

$$\left(\frac{\partial^2 \mathcal{L}}{\partial \varphi_j \partial \varphi_i}\right)_{\tilde{\varphi}} \frac{d\tilde{\varphi}_j}{d\delta} + \left(\frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial \varphi_i}\right)_{\tilde{\varphi}} = 0 \tag{3}$$

or

$$\frac{d\tilde{\varphi}_i}{d\delta} = [M^2]^{-1}_{ij}\left(\frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial \varphi_j}\right)_{\tilde{\varphi}} = M^{-2}_{ii}\left(\frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial \varphi_i}\right)_{\tilde{\varphi}} \tag{4}$$

where M^2_{ij} is the $(\text{mass})^2$ matrix, assumed here to have been transformed to the diagonal representation. For normal fields, $\tilde{\varphi}_i \equiv 0$, (4) requires $\left(\frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial \varphi_i}\right)_{\tilde{\varphi}} = 0$. For the fields we are interested in, $\tilde{\varphi}_i(\delta) \neq 0$, $M^2_{ii} = 0 \Leftrightarrow$ a singularity in $\frac{d\tilde{\varphi}_i(\delta)}{d\delta}$, unless the singularity of M^{-2}_{ii} is canceled by a zero of $\left(\frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial \varphi_i}\right)_{\tilde{\varphi}}$ or, conversely, the singularity in $\frac{d\tilde{\varphi}_i(\delta)}{d\delta}$ is carried by $\left(\frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial \varphi_i}\right)_{\tilde{\varphi}}$. Intuitively, the singularity in $\frac{d\tilde{\varphi}_i}{d\delta}$ associated with a zero of M^2_{ii} occurs when the theory is about to become unstable as a $(\text{mass})^2$ passes from a positive to a negative value.

More specifically, suppose we characterize the leading behavior of a vacuum solution[15] $\tilde{\varphi}_o(\delta)$ near a possible singular point $\bar{\delta}$ by the power law (i) $\tilde{\varphi}_o(\delta) = \alpha + \beta(\delta-\bar{\delta})^{\eta}$ with $\eta > 0$ or (ii) $\tilde{\varphi}(\delta) = \beta(\delta-\bar{\delta})^{\eta}$, $\eta \neq 0$, which includes the root, pole, or regular behavior possible in polynomial Lagrangians. For case (i) $\alpha \neq 0$, $\eta > 0$, we have $\varphi'_o(\delta) = \left((\delta-\bar{\delta})^{\eta-1}\right)$. With $\mathcal{L} = \mathcal{L}_{sym} + \delta\mathcal{L}_B$, $\left(\frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial \varphi_o}\right)_{\tilde{\varphi}_o} = \left(\frac{\partial \mathcal{L}_B}{\partial \varphi_o}\right)_{\tilde{\varphi}_o} \approx \sum_{p=o}^{N-1} C_p(\delta-\bar{\delta})^{\eta p}$, where N is the highest power of φ_o in $\mathcal{L}_B$, where $\mathcal{L}_B$ is the symmetry breaking part of $\mathcal{L}$. If $C_o \neq 0$, and this is always the case if only linear breaking is allowed so that operator PCAC conditions are exactly implemented, then a zero of M^2_{oo} requires a singular $\tilde{\varphi}'_o$ and conversely. Otherwise $M^2_{oo} = (\delta-\overline{\delta})^{(\overline{p}-1)\eta+1}$, where $\overline{p}$ is the lowest p for which $C_p \neq 0$, and M^2_{oo} vanishes ($\eta > 0$) irrespective of η but a singularity of φ'_o is not implied. For case (ii), Eq. (4) requires $M^2_{oo} = (\delta-\overline{\delta})^{\eta(q-2)+1}$ where q is the lowest (highest) power of φ_o in the expansion of $\mathcal{L}_B(\tilde{\varphi})$ when $\eta > 0$ ($\eta < 0$). For the interesting cases q = 1,2,3,4 the requirement $M^2_{oo} \rightarrow 0$ is equivalent to $\eta < 1$, η unrestricted, $\eta \geqslant -1$, $\eta > -\frac{1}{2}$ respectively. Thus with only linear breaking, $\eta < 1$ and φ'_o is necessarily singular at $M^2_{oo} = 0$, and conversely. $M^2_{oo} \rightarrow$ constant requires $\eta = 1$, $\eta = \infty$, $\eta = -1$, $\eta = -\frac{1}{2}$ for q = 1,2,3,4 respectively. For q = 1, $M^2_{oo} \rightarrow (\delta-\overline{\delta})^{1-\eta}$ and poles of $\tilde{\varphi}_o$ are associated with zeros of M^2_{oo}, a regular $\tilde{\varphi}_o$ with a non zero mass, and root-type behavior of $\tilde{\varphi}_o$ with root-type vanishing of M^2_{oo} for $\eta < 1$. For q = 2, $M^2_{oo} \rightarrow 0(\delta-\overline{\delta})$ independent of η and there is no necessary connection. For q = 3 and 4, the analysis proceeds in a similar manner.

To summarize, when the Lagrangian symmetry breaking is linear in fields, a vanishing mass of one of these fields implies a singularity in its vacuum expectation value and conversely. When higher powers of the fields are contained in the symmetry breaking part of the Lagrangian, it may be possible to avoid the connection between singularities and mass zeros.

In the following section, we study a number of simple models. The emphasis is placed on models which have symmetry breakers with two or more powers of the basic fields because this type of breaking has received comparatively little attention in the literature and because cases in which zero mass $\not\Leftrightarrow$ singularity can be studied in detail. The SU(2) models are, of course, for illustration only, and no physical applications are implied.

III. Singularities in SU(2)-, Chiral SU(3)-, and Scale Breaking Models.

The following examples illustrate the phenomenon discussed in Sec. II. Let $\varphi_{1,2,3}$ be scalar fields with $\sum_i \varphi_i \varphi_i \equiv \varphi^2$. Then with

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i - \beta(\phi^2 - a^2)^2 + \delta(\phi_3^2 + \rho\phi^2\phi_3) = \mathcal{L}_S + \delta\mathcal{L}_B \tag{5}$$

where $\mathcal{L}_S$ has SU(2) symmetry and $\mathcal{L}$ and $\mathcal{L}_B$ have I_3 symmetry. The extremum condition has the Goldstone solution

$$\tilde{\varphi}_1 \equiv 0, \ \tilde{\varphi}_2 \equiv 0$$

$$\tilde{\varphi}_3 = \frac{3\rho}{8\beta}\delta + \tfrac{1}{2}\left[\left(\frac{3\rho\delta}{4\beta}\right)^2 + 4\left(\frac{\delta}{2\beta} + a^2\right)\right]^{\frac{1}{2}} \tag{6}$$

which is singular at $\overline{\delta}$, where

$$\overline{\delta}: \quad \left(\frac{3\rho}{4\beta}\delta\right)^2 + 4\left(\frac{\delta}{2\beta} + a^2\right) = 0 \tag{7}$$

$$\tilde{\varphi}_3(\overline{\delta}) = \frac{3\rho}{8\beta}\overline{\delta} \tag{8}$$

The mass matrix is diagonal with

$$M_{11}^2 = M_{22}^2 = \delta(\rho\tilde{\varphi}_3 + \tfrac{1}{2})$$

$$M_{33}^2 = 3\rho\,\delta\tilde{\varphi}_3 + 4\delta + 8\beta a^2 \tag{9}$$

$M_{11}^2 = M_{22}^2 \to 0$ as $\delta \to 0$ as required by the Goldstone theorem and $M_{33}^2(\overline{\delta}) = 0$ as required by (4) because $\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_3}\right)_{\tilde{\varphi}_3} = 2\varphi_3 + 3\rho\varphi_3^2$ does not in general vanish at the singular point. The special case $\rho = 0$, $\overline{\delta} = -2\beta a^2$, $\tilde{\varphi}_3(\overline{\delta}) = 0$ behaves like

$$\tilde{\varphi}'_3 = 0\left((\delta-\bar{\delta})^{-\frac{1}{2}}\right)$$

$$M^2_{33} = 0(\delta-\bar{\delta})$$

$$\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_3}\right)_{\tilde{\varphi}} = 0\left((\delta-\bar{\delta})^{\frac{1}{2}}\right) \tag{10}$$

so that near $\bar{\delta}$, (4) requires again a zero mass because $\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_3}\right)_{\tilde{\varphi}}$ does not diverge or, conversely (4) requires a singularity in $\varphi_3(\delta)$ because $\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_3}\right)_{\tilde{\varphi}}$ does not cancel the zero of M^2_{33}. The interesting special case $\bar{\delta} = 0$, $a^2 = 0$ with $\tilde{\varphi}_3(\bar{\delta}) = 0$ satisfies (4) in exactly the same way (see below). That exceptional cases exist in which a singularity of $\tilde{\varphi}(\delta)$ does not imply a vanishing scalar mass is illustrated by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \beta(\phi^2-a^2)^2 + \delta\varphi_3^4 \tag{11}$$

which has a Goldstone solution $\tilde{\varphi}_1 \equiv 0$, $\tilde{\varphi}_2 \equiv 0$, and $\tilde{\varphi}_3(\delta) = \sqrt{\frac{\beta a}{\beta-\delta}}$ with <u>both</u> $\frac{d\varphi_3(\delta)}{d\delta}$ and $\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_3}\right)_{\tilde{\varphi}}$ of $0\left((\delta-\bar{\delta})^{-\frac{3}{2}}\right)$ near $\bar{\delta} = \beta$ but with $M^2_{33} = 4\beta a^2$, independent of δ.

Let us now consider the implications of (4) for $\varphi_i(\delta) \neq 0$ realizations of scale symmetry. In this case, $|M^2| \to 0$, whether the scale symmetry is realized conventionally or in the Goldstone manner, when the scale breaking (proportional to δ) is shut off. Thus $|M^2|$ is necessarily zero at $\delta = 0$, requiring $\tilde{\varphi}_i$ to be singular at the symmetry point unless $\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_i}\right)_{\tilde{\varphi}_i}$ carries the singularity. Consider for example the Lagrangian (5) with $a^2 = 0$. Then $\mathcal{L}$ is scale invariant when $\delta = 0$. A spontaneous broken solution is $\tilde{\varphi}_1 \equiv 0$, $\tilde{\varphi}_2 \equiv 0$, and

$$\tilde{\varphi}_3 = \frac{3\rho\delta}{8\beta} \pm \tfrac{1}{2}\left[\left(\frac{3\rho\delta}{4\beta}\right)^2 + \frac{4\delta}{2\beta}\right]^{\frac{1}{2}}$$

which has a branch point at $\delta = \overline{\delta} = 0$ where $\tilde{\varphi}_3(\overline{\delta}) = 0$. Here

$$M_{33}^2 = 0(\delta)$$

$$\frac{d\tilde{\varphi}_3}{d\delta} = 0(\delta^{-\frac{1}{2}})$$

$$\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_3} = 0(\delta^{\frac{1}{2}}) \tag{12}$$

so that the zero of M_{33}^{-2} (where φ_3 is coupled linearly to θ^{μ}_{μ}, the energy-momentum stress tensor) requires a singular $\varphi_3(\delta)$ and conversely.

That the necessary singularity in M_{33}^{-2} can be canceled, by careful planning, is illustrated by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial^{\mu}\varphi_i\partial_{\mu}\varphi_i - \beta\varphi^4 + \delta\varphi_3^3 \tag{13}$$

which has an asymmetric vacuum $\tilde{\varphi}_1 \equiv 0$, $\tilde{\varphi}_2 \equiv 0$, $\tilde{\varphi}_3 = \frac{3\delta}{4\beta}$ with

$$M_{33}^2 = 0(\delta^2)$$

$$\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial\varphi_3}\right)_{\tilde{\varphi}} = 0(\delta^2) \tag{14}$$

In this case, both scale symmetry and SU(2) symmetry breakdown is realized smoothly in the symmetry breaking parameters.

We now discuss the SU(3) σ-models in the light of this analysis. The basis fields are the eighteen scalars and pseudoscalars (u_i, v_i), $i = 0,\ldots 8$, which span a $(3,3^*)\oplus(3^*,3)$ representation of SU(3) X SU(3). With I_2, I_3^+ and I_4 the usual[16] bilinear, trilinear (positive parity), and quartic scalar SU(3)XSU(3) invariants, the original model of Lévy[8] restricted to SU(3)XSU(3) [but not SU(2)] linear breaking is

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu u_i \partial_\mu u_i + \tfrac{1}{2}\partial^\mu v_i \partial_\mu v_i - \alpha_1 I_2 - \alpha_2 I_2^2 - \beta I_3^+ - \gamma I_4 + \delta u_o \qquad (15)$$

The extremum conditions have a solution $\tilde{u}_i(\delta) = \tilde{u}_o(\delta)\delta_{io}$, $\tilde{v}_i \equiv 0$ where

$$\tilde{u}_o: \quad 0 = \frac{\partial\mathcal{L}}{\partial u_o} = -u_o^3\left(4\alpha_2 + \frac{16\gamma}{3}\right) - u_o^2\, 4\sqrt{6}\beta - u_o(2\alpha_1) + \delta \qquad (16)$$

With $\alpha_1 = 4\beta^2/(\alpha_2 + 4/3\gamma)$, $\tilde{u}_o(\delta)$ has a branch point at $\bar{\delta} = -1/27\left(\sqrt{6}\beta/(\alpha_2 + 4/3\gamma)\right)^3$ where $\tilde{u}_o(\bar{\delta}) = -\sqrt{6}\beta/3(\alpha_2 + 4/3\gamma) = +\sqrt[3]{\bar{\delta}}$. At this branch point $\left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o} = 1$ so that (4) requires a singularity of M_{oo}^{-2}, which is indeed the case, as a simple calculation will verify.

Suppose we now drop the βI_3^+ and δu_o terms in (16) and consider the approach to the scale limit $\alpha_1 = 0$. Then we have a solution $\tilde{u}_o(\alpha_1) = \pm\frac{1}{2}\sqrt{\frac{-2\alpha_1}{\alpha_2+4/3\gamma}}$ with a $\sqrt{\alpha_1}$ branch point. Here $\left(\frac{\partial}{\partial\alpha_1}\frac{\partial\mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o} = 0(\sqrt{\alpha_1})$ so that (4) requires M_{oo}^2 to vanish like $0(\alpha_1)$. In this case, the scale limit is accompanied by a singularity of the theory.

Again, special cases exist in which the zero of M_{oo}^2 does not require a singularity of $\tilde{u}_o$. Consider for example the case $\alpha_1 = 0$ in (15), discussed extensively by Carruthers[17] and Carruthers and Haymaker.[4-6] With no chiral SU(3)XSU(3) breaking ($\delta = 0$), we have a solution $\tilde{u}_i(\beta) = \delta_{io}\tilde{u}_o(\beta)$, $\tilde{v}_i(\beta) \equiv 0$, with $\tilde{u}_o(\beta)$ a solution of

$$u_o^2\left(2\sqrt{6}\beta + (\alpha_2 + 4/3\gamma)u_o\right) = 0 \qquad (17)$$

As pointed out by Carruthers and Haymaker,[6] the $\tilde{u}_o \equiv 0$ solutions are unstable against chiral perturbations.[18] For $\tilde{u}_o(\beta) = -2\sqrt{6}\beta/(\alpha_2+4/3\gamma) = 0(\beta)$, we have a non-singular theory because $M_{oo}^2 = 0(\beta^2)$ (which never develops a negative $(\text{mass})^2$) and $\left(\frac{\partial}{\partial\beta}\frac{\partial\mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o} = 0(\beta^2)$ so that the necessary zero of M_{oo}^2 in the scale limit is canceled by $\left(\frac{\partial}{\partial\beta}\frac{\partial\mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o}$.

We stress that the scale limit, because of (4), is a delicate one which, barring cancelations, is singular in the scale breaking parameter.

This analysis of the relation between singularities and mass zeros was motivated by and has been a useful guide in a numerical study, in tree approximation, of spontaneously broken solutions of Lagrangian (15) and its generalizations. In the simplest cases $\left.\frac{\partial^2 \mathcal{L}}{\partial \varphi_o{}^2}\right|_{\tilde{\varphi}_o} = 0$ is equivalent to the statement that $\tilde{\varphi}_o$ is a multiple root of the extremum condition if $\mathcal{L}$ is a polynomial. The analysis is not, however, restricted to polynomial Lagrangians, but is equally applicable to systems like

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i - e^{\varphi^2} + \alpha\varphi^2 + \delta\varphi_3{}^2 \tag{18}$$

with a ground state solution $\tilde{\varphi}_3 = \sqrt{\ln(\alpha+\delta)}$ and $M_{11}^2 = M_{22}^2 = 2\delta$, $M_{33}^2 = 4(\alpha+\delta)\ln(\alpha+\delta)$. It is unclear whether such branch points are related to those of Li and Pagels.[3] We are currently studying this question in the chiral SU(3) context.

We close by discussing several cases in which the expression (4) itself fails because of an implicit dependence of $\mathcal{L}$ on δ, not only through the fields φ_i, but also through the coefficients of the various terms in $\mathcal{L}$. For example, with

$$\mathcal{L} = \mathcal{L}_{kin} - \alpha\, I_2 - \beta I_3^{\pm} - \gamma I_4 + \delta[a(u_o + cu_8) + d(U_o + CU_8)] \tag{19}$$

where U_o and U_8 are the $(3,3^*)\oplus(3^*,3)$ decomposition of the tensor product of the basic multiplet with itself,[19] we have a solution $\tilde{u}_i(\delta) = \tilde{u}_o(\delta)\delta_{io}$, $\tilde{v}_i \equiv 0$, provided the constraint

$$\delta\left(ca - 4\sqrt{\tfrac{2}{3}}\, dC\, \tilde{u}_o\right) = 0 \tag{20}$$

is maintained, with $\tilde{u}(\delta)$ a solution of

$$0 = 2\alpha u_o + 4\sqrt{6}\beta u_o^2 + \frac{16\gamma}{3} u_o^3 - \delta a - 8\delta d\sqrt{\tfrac{2}{3}}\, u_o \qquad (21)$$

These two equations can be regarded as determining $\tilde{u}_o(\delta)$ and $a(\delta)$. Thus the form (4) is modified to

$$\frac{d}{d\delta}\left(\frac{\partial \mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o} = 0 = \left(\frac{\partial^2 \mathcal{L}}{\partial u_o^2}\right)_{\tilde{u}} \frac{d\tilde{u}_o}{d\delta} + \left(\frac{\partial \mathcal{L}}{\partial a}\frac{da}{d\delta} + \frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o} \qquad (22)$$

or

$$\frac{d\tilde{u}_o}{d\delta} = \frac{\left(\frac{\partial \mathcal{L}}{\partial a}\frac{da}{d\delta} + \frac{\partial}{\partial \delta}\frac{\partial \mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o}}{M_{oo}^2} \qquad (23)$$

The singularity analysis then proceeds as above, but from (23) rather than (4). (In this Lagrangian, for example, there is a spontaneously broken solution in which $M_{oo}^2 \neq 0$ at the branch point of $\tilde{u}_o$, and $\tilde{u}_o$ does not have a singularity at $M_{oo}^2 = 0$.) Consider now the situation in which scale symmetry and chiral symmetry are realized simultaneously, that is

$$\mathcal{L} = \mathcal{L}_{kin} - \alpha_2 I_2^2 - \gamma I_4 + \delta[a(u_o + cu_8) + d(U_o + CU_8) + eG_8] \qquad (24)$$

where for generality we have included a $(1,8)\oplus(8,1)$ non-pole term $G_8 = d_{8ij}(u_i u_j + v_i v_j)$. The extremum equation is

$$\frac{\partial \mathcal{L}}{\partial u_o} = 0 = -u_o^3\left(4\alpha_2 + \frac{16\gamma}{3}\right) + \delta a + 8\sqrt{\tfrac{2}{3}}\,\delta d u_o \qquad (25)$$

subject to the constraint

$$\delta\left(ca - 4\sqrt{\tfrac{2}{3}}\,Cdu_o + 2e\sqrt{\tfrac{2}{3}}\,u_o\right) = 0 \qquad (26)$$

which is necessary to maintain $\left.\frac{\partial \mathcal{L}}{\partial u_8}\right|_{\tilde{u},\tilde{v}} = 0$. Thus, $a = u_o c^{-1}[4\sqrt{\tfrac{2}{3}}Cd - 2\sqrt{\tfrac{2}{3}}e]$ and we have the solutions

(i) $4\alpha_2 + 16\gamma/3 = 0$, $\tilde{u}_o = -a/8\sqrt{\frac{2}{3}}d = ca/2\sqrt{\frac{2}{3}}(2dc-e)$

(ii) $4\alpha_2 + 16\gamma/3 \neq 0$, $\tilde{u}_o \equiv 0$, $a = 0$

(iii) $4\alpha_2 + 16\gamma/3 \neq 0$, $\tilde{u}_o \neq 0$,

$$\tilde{u}_o = \left\{2\delta\sqrt{\tfrac{2}{3}}[d(\frac{2C}{c} + 4) - \frac{e}{c}]\right\}^{\frac{1}{2}} \tag{27}$$

In special case (i), there is no unique C-parameter, $\tilde{u}_o$ is a non-zero constant and (23) is satisfied by $M^2_{oo} = 0(\delta)$ and $\frac{\partial\mathcal{L}}{\partial a}\frac{da}{d\delta} + \left(\frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial u_o}\right)_{\tilde{u}_o} \equiv 0$, $\frac{d\tilde{u}_o}{d\delta} \equiv 0$. Special case (ii) goes through similarly. The more interesting case (iii) satisfies (23) in the form $M^2_{oo} = 0(\delta)$, $\frac{d\tilde{u}_o}{d\delta} = 0(\delta^{-\frac{1}{2}})$ and $\frac{\partial\mathcal{L}}{\partial a}\frac{da}{d\delta} + \frac{\partial}{\partial\delta}\frac{\partial\mathcal{L}}{\partial u_o} = 0(\sqrt{\delta})$, that is, the mass zero induces again a singularity of the theory. Note that solution (iii) is singular at $c = 0$, $C \neq 0$. The following expressions isolate the effects of the bilinear breaking terms on the pseudoscalar octet (ps) masses. With $c = C$, $e \neq o$

$$(M^2_{ij})_{ps} = \delta[2(4dc-e)/c]\{\sqrt{\tfrac{2}{3}}\delta_{ij} + c\, d_{8ij}\} \tag{28}$$

and with $c \neq C$, $e = 0$

$$(M^2_{ij})_{ps} = 8\delta d\{\sqrt{\tfrac{2}{3}}[\tfrac{1}{2}(\frac{C}{c} + 2) - \tfrac{1}{2}]\delta_{ij} + C\, d_{8ij}\} \tag{29}$$

It is clear that, so far as the pseudoscalar masses are concerned, the effects of bilinear $(1,8)\oplus(8,1)$ breaking terms (non-pole) are largely invisible. This can not be said for the bilinear $(3,3^*)\oplus(3^*,3)$ breaking term, which is not surprising because the bilinear U_i, V_i have terms linear in the fields u_i, v_i when expanded about the ground state solution.[9]

Finally, we wish to mention a variation of the Lagrangian of Eq. (24)

which can be used to illustrate the scheme recently proposed by Mathur.[20] We write

$$\mathcal{L} = \mathcal{L}_{kin} - \alpha_2 I_2^2 - \gamma I_4 + \delta[a(u_o + cu_8) + d(U_o + cU_8) + I_2] \tag{30}$$

which can be written in the form proposed by Mathur,[20] where δI_2 breaks scale invariance but not chiral invariance and $\delta \to 0$ produces the combined scale- and chiral-invariant limit. (Note that the η'-π degeneracy,[21] which afflicts the linear breaking model if I_3^+ is absent, is lifted here by the bilinear breaking terms.) Requiring $\tilde{u}_8 \equiv \langle u_8 \rangle = 0$, we have a solution $\tilde{u}_i(\delta) = \tilde{u}_o(\delta)\ \delta_{io}$, $\tilde{v}_i = 0$ with a contraint

$$\delta c(a - 4\sqrt{\tfrac{2}{3}}\, d\tilde{u}_o) = 0 \tag{31}$$

where $\tilde{u}_o(\delta)$ is a solution of

$$0 = (4\alpha_2 + \frac{16\gamma}{3})\tilde{u}_o^3 - 2\delta\tilde{u}_o - \delta a - 8\delta d\sqrt{\tfrac{2}{3}}\tilde{u}_o. \tag{32}$$

When $4\alpha_2 + \frac{16\gamma}{3} = 0$, a possible solution to Eqs. (31) and (32) is the following:

$\tilde{u}_o = -\frac{3}{2}\, a$, independent of δ

$d = -\frac{1}{6}\sqrt{\frac{3}{2}}$

$c \neq 0$

$$(M_{oo}^2)_{scalar} \sim 0(\delta), \qquad (M_{ii}^2)_{ps} \sim 0(\delta) \quad i = 1 \ldots 8 \tag{33}$$

$$(M_{oo}^2)_{ps} \sim 0(\delta), \qquad (M_{ii}^2)_{scalar} \sim 0(1) \quad i = 1 \ldots 8.$$

Now since

$$(M^2_{ii})_{ps} = 8\delta d\{\sqrt{\tfrac{2}{3}} + cd_{8ii}\},\ i = 1\ldots 8$$

$$= -2\sqrt{\tfrac{2}{3}}\delta\ \{\sqrt{\tfrac{2}{3}} + cd_{8ii}\}, \qquad (34)$$

the usual value for the c-parameter[22]

$$c = \sqrt{2}\,\frac{(M_\pi^2 - M_k^2)}{M_\pi^2 + M_k^2} \approx -1.25 \qquad (35)$$

is found. This model contains the basic features of the Mathur scheme and, as a result of the form of the constraint equation, has a vacuum solution with trivial dependence on δ, insuring a smooth scale limit which avoids the singularities which usually accompany the necessary zero mass in the scale limit.

Investigation of extended models of this kind, especially with regard to expansions in symmetry breaking parameters and smoothness of matrix elements of symmetry breaking parts of $\mathcal{L}$ in pseudoscalar states is in progress. For a discussion of how low mass scalar poles necessarily interfere with such smoothness in a current algebra approach with pole saturation, and a motivation for considering such models, see Ref. (23).

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Footnotes and References

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15. We now consider the case in which only one of the fields, φ_o, has a non-vanishing vacuum expectation value.

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