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**ZERO RELAXATION AND DISSIPATION LIMITS
FOR HYPERBOLIC CONSERVATION LAWS**

By

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Zero Relaxation and Dissipation Limits for Hyperbolic Conservation Laws

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Abstract

We are interested in hyperbolic systems of conservation laws with relaxation and dissipation, particularly the zero relaxation limit. Such a limit is of interest in several physical situations, including gas flow near thermo-equilibrium, kinetic theory with small mean free path, and viscoelasticity with vanishing memory. In this article we study hyperbolic systems of two conservation laws with relaxation. For the stable case where the equilibrium speed is subcharacteristic with respect to the frozen speeds, we illustrate for a model in viscoelasticity that no oscillation develops for the nonlinear system in the zero relaxation limit. For the marginally stable case where the equilibrium speed may equal one of the frozen speeds, we show for a model in phase transitions that no oscillation arises when the dissipation is present and goes to zero more slowly than the relaxation. Our analysis includes the construction of suitable entropy pairs to derive energy estimates. We need such energy estimates not only for the compactness properties but also for the deviation from the equilibrium of the solutions for the relaxation systems. The theory of compensated compactness is then applied to study the oscillation in the zero relaxation limit.

1. Introduction

Consider the hyperbolic system of conservation laws with relaxation

$$(1.1) \quad \begin{cases} u_t + f(u, v)_x = 0, \\ v_t + g(u, v)_x = \frac{V_*(u) - v}{\tau(u)}. \end{cases}$$

The first equation is a conservation law, while the second equation contains a relaxation mechanism with $V_*(u)$ as the equilibrium value for v and $\tau(u)$ the relaxation time [13]. We are interested in the zero relaxation limit $\tau(u) \rightarrow 0$. In many physical situations, certain stability criteria hold, no oscillation is expected to develop in the limit, and the solution converges to that of the equilibrium hyperbolic conservation law

$$(1.2) \quad u_t + f(u, V_*(u))_x = 0.$$

The relaxation phenomenon is important in many physical situations: in the kinetic theory the relaxation time is the mean free path, and in viscoelasticity the strength of memory. In gas dynamics the phenomenon occurs when the gas is in the thermo-nonequilibrium. It also occurs in river flow and traffic flow (see [21]). The zero relaxation limit reduces the number of equations and is more singular than other limits such as the zero dissipation limit. Shock, initial layers, and boundary layers can develop.

There is a stability criterion for (1.1) under which (1.2) is expected to be the limiting equation. The criterion can be obtained through asymptotic expansion of the Chapman-Enskog type in kinetic theory [13]. The first order expansion of (1.1) is obtained by setting v to be the equilibrium, $v = V_*(u)$. The resulting equation (1.2) is used in the second order expansion only to the extent of setting the primary characteristic direction by the equilibrium characteristic speed λ_* :

$$(1.2)' \quad \partial_t + \lambda_*(u)\partial_x \cong 0,$$

$$(1.3) \quad \lambda_*(u) \equiv \frac{d}{du} f(u, V_*(u)).$$

For the second order approximation, we set v_1 to be the deviation from the equilibrium and obtain from the second equation in (1.1)

$$\begin{cases} v = V_*(u) + v_1, \\ v_1 \equiv \tau(u)(v_t + g(u, v)_x) \\ \cong \tau(u)(V_*(u)_t + g(u, V_*(u))_x). \end{cases}$$

Here we have made an a priori hypothesis that v_1 is small and that its gradients are even smaller as a result of the dissipation induced by the relaxation. Using (1.2)', we obtain

$$v_1 \cong \tau(u)(-\lambda_*(u)V_*'(u) + g_u(u, V_*(u)) + V_*'(u)g_v(u, V_*(u)))u_x.$$

Inserting this into the first equation (1.1), we obtain the second order viscous approximation to (1.1):

$$(1.4) \quad u_t + f(u, V_*(u))_x = (\beta(u)u_x)_x.$$

The viscosity $\beta(u)$ is related to the equilibrium speed λ_* , and the frozen speeds λ_1 and λ_2 , characteristic speeds of (1.1), as follows:

$$(1.5) \quad \begin{cases} \beta(u) = \tau(u)(\lambda_*(u) - \lambda_1(u, V_*(u)))(\lambda_2(u, V_*(u)) - \lambda_*(u)), \\ \lambda_1(u, v) = \frac{1}{2}(f_u + g_v) - \frac{1}{2}((f_u - g_v)^2 + 4f_v g_u)^{\frac{1}{2}}, \\ \lambda_2(u, v) = \frac{1}{2}(f_u + g_v) + \frac{1}{2}((f_u - g_v)^2 + 4f_v g_u)^{\frac{1}{2}}. \end{cases}$$

The stability criterion is that the viscosity $\beta(u)$ is positive and, equivalently, the equilibrium speed λ_* is subcharacteristic with respect to the frozen speeds λ_1 and λ_2 :

$$(1.6) \quad \lambda_1 < \lambda_* < \lambda_2.$$

Our first goal is to show that, in the stable case (1.6), no oscillation develops in the zero dissipation limit and that the solutions of (1.1) converge to those of (1.2). We show this fact for a model in viscoelasticity [11]:

$$(1.7) \quad \begin{cases} u_t + \sigma_x = 0, \\ (\sigma - f(u))_t + \frac{\sigma - \mu f(u)}{\delta} = 0, \quad f(0) = 0. \end{cases}$$

Condition (1.6) holds when $0 < \mu < 1$, $f'(u) > 0$. This prevents the creeping phenomenon of Maxwell.

Our second goal is to show that when (1.4) is marginally stable, that is,

$$(1.8) \quad \lambda_1 \leq \lambda_* \leq \lambda_2,$$

then the oscillation can be suppressed with the additional effect of dissipation. For this we study a model in phase transitions:

$$(1.9) \quad \begin{cases} u_t + \left(\frac{1 + (\mu - 1)v}{u} \right)_x = 0, \\ v_t = \frac{V_*(u) - v}{\delta}, \mu > 1, \delta > 0, \end{cases}$$

where

$$V_*(u) = \begin{cases} 0, & \text{for } 0 < u < 1, \\ \frac{u-1}{\mu-1}, & \text{for } 1 < u < \mu, \\ 1, & \text{for } \mu < u. \end{cases}$$

The model shares the same essential qualitative behavior as the model for polymorphic phase transitions proposed in [18,10].

We show that the solutions of the viscous equations

$$(1.10) \quad \begin{cases} u_t + \left(\frac{1 + (\mu - 1)v}{u} \right)_x^2 = \varepsilon u_{xx}, \\ v_t = \frac{V_*(u) - v}{\delta} + \varepsilon v_{xx} \end{cases}$$

converge to the equilibrium solutions when δ and ε tend to zero, but the dissipation dominates $\delta\varepsilon^{-1} \rightarrow 0$.

The main part of our analysis consists of energy estimates for the boundedness and compactness, and the deviation from the equilibrium for the solutions of (1.1). We also need to estimate the extent to which a solution of (1.1) satisfies the entropy inequality for (1.2) so as to ensure the admissibility of the limit solutions for (1.2). These are achieved by properly chosen entropy functions. We then apply the theory of compensated compactness to control the oscillation and to obtain convergence.

In the next section we discuss preliminaries for the models (1.7) and (1.9). Section 3 concerns the construction of entropy pairs for (1.1) and relates them to those for (1.2). In Section 4 we briefly review the theory of compensated compactness as needed for our study. Our approach is general in the sense that it applies to two general conservation laws with relaxation. We use the particular models (1.7) and (1.9) because both have invariant regions, which yield a priori sup norm estimates as shown in Section 5. This allows us to establish the existence of L^∞ solutions. Finally, in Sections 7 and 8, we study the convergence for (1.7) and (1.9), respectively, in the zero relaxation limits. We note that, since both (1.7) and (1.9) contain a linearly degenerate field in the sense of Lax [12], one might expect the oscillation to persist [4]. The reason why oscillation does not persist is due to the dissipation effect of relaxation. Our analysis makes use of this fact by first studying the convergence of u and, through the coupling, the convergence of v . The convergence of v is not uniform near the time zero because of the initial layer that is expected to develop.

Relaxation is present for many physical models in gas dynamics, elasticity, kinetic theory, multiphase flow, and phase transition. It would be interesting to study the zero relaxation limit for these models. For qualitative studies of nonlinear waves for relaxation models, see [13], and for the zero relaxation limit with particular data, see [2,7].

2. Preliminaries

The elastic model (1.7) can be rewritten as

$$(2.1) \quad \begin{cases} u_t + (f(u) - v)_x = 0, \\ v_t = \frac{(1 - \mu)f(u) - v}{\delta}, \quad v \equiv \sigma - f(u), \quad \mu > 0, \quad \delta > 0. \end{cases}$$

The corresponding equilibrium equation is

$$(2.2) \quad u_t + (\mu f(u))_x = 0.$$

The characteristic speeds λ_1 and λ_2 of (2.1) are called the frozen speeds, and that λ_* of (2.2) the equilibrium speed:

$$(2.3) \quad \lambda_1 = 0, \quad \lambda_2 = f'(u), \quad \lambda_* = \mu f'(u).$$

The basic assumption for the model (2.1) is

$$(2.4) \quad 0 < \mu < 1, \quad f'(u) > 0,$$

which yields the stability criterion

$$(2.5) \quad \lambda_1 < \lambda_* < \lambda_2.$$

The equilibrium equation for the phase transition model (1.9) is

$$(2.6) \quad u_t + f_*(u)_x = 0,$$

where

$$f_*(u) = \begin{cases} \frac{1}{u^2}, & 0 < u < 1, \\ 1, & 1 < u < \mu, \\ \frac{\mu^2}{u^2}, & u > \mu. \end{cases}$$

The frozen speeds λ_1 and λ_2 , and the equilibrium speed λ_* are

$$(2.7) \quad \begin{cases} \lambda_1 = -\frac{2(1 + (\mu - 1)v)^2}{u^3}, \quad \lambda_2 = 0, \\ \lambda_* = f'_*(u) = \begin{cases} \frac{2}{u^3}, & u < 1, \\ 0, & 1 < u < \mu, \\ \frac{-2\mu^2}{u^3}, & u > \mu. \end{cases} \end{cases}$$

Thus, for the equilibrium states $v = V_*(u)$, the strict stability criterion (1.6) does not hold. Indeed

$$(2.8) \quad \begin{cases} \lambda_*(u) = \lambda_1(u, V_*(u)), \text{ for } u < 1 \text{ and for } u > \mu, \\ \lambda_*(u) = \lambda_2(u, V_*(u)) = 0, \text{ for } 1 < u < \mu, \\ \lambda_1(u_1, V_*(u)) \leq \lambda_*(u) \leq \lambda_2(u, V_*(u)). \end{cases}$$

This weak stability property prompts us to introduce the dissipation into the model (2.6) to stabilize the process of the zero relaxation limit.

3. Entropy

This section is concerned with the entropy pairs (η, q) for our models (2.1) and (1.9). It is well known (see [12]) that, for the general conservation laws

$$(3.1) \quad U_t + F(U)_x = G(U), \quad U \in \mathbb{R}^n,$$

the entropy pairs satisfy

$$(3.2) \quad \nabla q = \nabla \eta \nabla F,$$

so that the smooth solutions U of (3.1) satisfy

$$\eta(U)_t + q(U)_x = \nabla \eta(U) \cdot G(U).$$

An admissible weak solution U of (3.1) satisfies the entropy condition

$$(3.3) \quad \eta(U)_t + q(U)_x \leq \nabla \eta(U) \cdot G(U),$$

for any convex entropy $\eta(U)$. For scalar conservation laws, a weak solution is admissible if and only if it satisfies (3.3) for any convex $\eta(U)$. For genuinely nonlinear systems, a weak solution is admissible if (3.3) holds for a convex entropy.

For the elastic model (2.1), we have from the compatibility condition for (3.2) that

$$\eta_{uu} + f'(u)\eta_{uv} = 0.$$

Thus the general representation of the entropy pairs is

$$(3.4) \quad \begin{cases} \eta(u, v; H, G) = \int^u H(f(\xi) - v) d\xi + G(v), \\ q(u, v; H) = \int^{f(u)-v} H(\xi) d\xi, \end{cases}$$

for any continuous functions H and G . The following ones are particularly useful:

$$(3.5) \quad \eta_0(u, v; H) = \int_{f^{-1}(\frac{v}{1-\mu})}^u H(f(\xi) - v) d\xi + \int^{f^{-1}(\frac{v}{1-\mu})} H(\mu f(\xi)) d\xi,$$

$$(3.6) \quad \bar{\eta}(u, v) = \int_{f^{-1}(\frac{v}{1-\mu})}^u (f(\xi) - v) d\xi + \mu \int^{f^{-1}(\frac{v}{1-\mu})} f(\xi) d\xi.$$

The desired property for η in (3.5) and (3.6) is that η_v vanishes at the equilibrium state $v = (1 - \mu)f(u)$ and that

$$(3.7) \quad \begin{cases} \bar{\eta}(u, v)_v \frac{v - (1 - \mu)f(u)}{\delta} \geq C_1 \frac{(v - (1 - \mu)f(u))^2}{\delta}, \\ \left| \eta_0(u, v)_v \frac{v - (1 - \mu)f(u)}{\delta} \right| \leq C_2 \frac{(v - (1 - \mu)f(u))^2}{\delta}, \end{cases}$$

for positive constants C_1 and C_2 , depending only on the function H and M defining the bounded set $\{(u, v) : 0 \leq v \leq f(u) \leq M\}$. The estimates (3.7) follow from a simple

calculation with the hypothesis that $f'(0) \gg 1$ so that $\bar{\eta}$ is convex. The usefulness of (3.7) follows from the fact that, for the model (2.1), the entropy inequality (3.3) takes the form

$$\eta(u, v)_t + q(u, v)_x \leq \eta(u, v)_v \frac{(1 - \mu)f(u) - v}{\delta},$$

and thus

$$\bar{\eta}(u, v)_t + \bar{q}(u, v)_x \leq -C_1 \frac{(v - (1 - \mu)f(u))^2}{\delta},$$

from (3.7). This yields an estimate for the deviation of the solutions of (2.1) from the equilibrium in Sections 6 and 7. The entropy functions $\eta_0(u, v; H)$ can become any entropy function $E(u)$ at the equilibrium:

$$(3.10) \quad \begin{cases} H(w) \equiv E'(f^{-1}\left(\frac{w}{\mu}\right)), \\ E(u) = \eta_0(u, (1 - \mu)f(u); H). \end{cases}$$

For the phase transition model (1.9), the entropy functions $\eta(u, v)$ satisfy

$$(\mu - 1)u\eta_{uu} + (1 + (\mu - 1)v)\eta_{uv} = 0.$$

Thus the general representation of the entropy pairs (η, q) is

$$(3.11) \quad \begin{cases} \eta(u, v; H, G) = \int^u H\left(\frac{1 + (\mu - 1)v}{\xi}\right) d\xi + G(v), \\ q(u, v; H) = \int^{\frac{1 + (\mu - 1)v}{u}} H(\xi) d\xi, \end{cases}$$

for any continuous functions H and G . A particular one

$$(3.12) \quad \bar{\eta}(u, v) = \frac{(1 + (\mu - 1)v)^2}{u} - 2(\mu - 1)v$$

has the desired property that, on any bounded set

$$\{(u, v) : 0 \leq v \leq 1, M^{-1}(1 + (\mu - 1)v) \leq u \leq M(1 + (\mu - 1)v)\},$$

$$(3.13) \quad \begin{cases} \nabla^2 \bar{\eta}(u, v) \geq 0, \\ \bar{\eta}(u, v)_v (v - V_*(u)) \geq C_0 (v - V_*(u))^2, \end{cases}$$

for any $C_0 > 0$, depending only on M . For the present case, the entropies $\eta(u, v; H, G)$ do not produce any convex entropy $E(u)$ for the equilibrium equation at the equilibrium $v = V_*(u)$. This prompts a different analytical approach in studying the zero relaxation and dissipation limits.

4. The Compensated Compactness Method

The theory of compensated compactness, originally introduced in [15,17,20] and a related observation in [1], is an efficient tool for studying the singular limit for conservation laws. In this section we review some results pertaining to our study.

Theorem 4.1([20]). Suppose that $U^\varepsilon : \mathbb{R}_+^2 \rightarrow \mathbb{R}^n$ is a sequence of bounded measurable functions

$$(4.1) \quad U^\varepsilon(x, t) \in K, \quad \text{a.e.},$$

for a bounded set K in \mathbb{R}^n and that, for function pairs (η_i, q_i) , $i = 1, 2$,

$$(4.2) \quad \eta_i(U^\varepsilon)_t + q_i(U^\varepsilon)_x \quad \text{compact in } H_{loc}^{-1}.$$

Then there exists a subsequence (still labeled U^ε) and Young measures

$$\nu_{x,t}(\lambda) \in \text{Prob}(\mathbb{R}^n),$$

such that

(i) for any continuous function g ,

$$w^* - \lim g(U^\varepsilon) = \langle \nu_{x,t}(\lambda), g(\lambda) \rangle = \int_n g(\lambda) d\nu_{x,t}(\lambda),$$

and

$$(4.3) \quad \langle \nu_{x,t}, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu_{x,t}, \eta_1 \rangle \langle \nu_{x,t}, q_2 \rangle - \langle \nu_{x,t}, \eta_2 \rangle \langle \nu_{x,t}, q_1 \rangle;$$

(ii) $U^\varepsilon(x, t) \rightarrow U(x, t)$ strongly if and only if $\nu_{x,t}$ is a Dirac mass

$$\nu_{x,t} = \delta_{U(x,t)},$$

for almost all (x, t) .

This theorem provides a framework by which one can prove strong convergence of the sequence $U^\varepsilon(x, t)$ satisfying (4.1) and (4.2) by deducing

$$\nu_{x,t}(\lambda) = \delta_{w^* - \lim U^\varepsilon(x,t)(\lambda)}$$

from the functional relation (4.3).

Theorem 4.2. Suppose $\nu = \nu_{x,t}$ satisfies (4.3) for entropy pairs $(\eta_i, q_i)(U)$, $i = 1, 2$, for the conservation laws (3.1), $U \in \mathbb{R}^n$.

(i) For the scalar laws ($n = 1$) with measure $\{U : F''(U) = 0\} = 0$, we set

$$\begin{cases} (\eta_1, q_1) = (U - k, F(U) - f(k)), \\ (\eta_2, q_2) = (F(U) - f(k), \int_k^U (F'(s))^2 ds), \end{cases} \quad k \text{ constant},$$

Then $\nu_{x,t} = \delta_{U(x,t)}$ a.e., and

$$\langle \nu_{x,t}, F(\lambda) \rangle = F(\langle \nu_{x,t}, \lambda \rangle).$$

- (ii) For two strictly hyperbolic conservation laws ($n = 2$) with the following property: For some $j \in \{1, 2\}$,

$$\nabla \lambda_j(U) \cdot r_j(U) \neq 0, \quad U \in \mathbb{R}^2,$$

or there exists a curve $u_j = V(u_i), j \neq i$, such that

$$(u_j - V(u_i))(\nabla \lambda_j(U) \cdot r_j(U)) > 0, \quad U \in \mathbb{R}^2 - \{u_j \neq V(u_i)\}.$$

Then

$$\text{supp } \nu \subset \{U : w_j(U) = \bar{w}_j\}, \quad \bar{w}_j \text{ some constant,}$$

provided that (4.3) is satisfied for all entropy pairs, where $\nabla f \cdot r_j = \lambda_j r_j, j = 1, 2$, and $w = (w_1, w_2)$ is the coordinate system of the Riemann invariants. In particular, if the system is genuinely nonlinear (i.e., $\nabla \lambda_j \cdot r_j \neq 0, j = 1, 2$), then $\nu_{x,t} = \delta_{U(x,t)}$.

Part (i) of Theorem 4.2 shows that two entropy pairs are sufficient to conclude the weak continuity of the flux function for the scalar conservation law [5] (see also [3,20]), while infinite pairs are needed for two conservation laws [9,19,16,3]. The following embedding theorems are useful for obtaining H^{-1} compactness needed in the above theorems.

Theorem 4.3 ([8]). Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $1 < q \leq p < r < \infty$. Then

$$(\text{compact set of } W_{loc}^{-1,q}(\Omega)) \cap (\text{bounded set of } W_{loc}^{-1,r}(\Omega)) \subset (\text{compact set of } W_{loc}^{-1,p}(\Omega)).$$

Theorem 4.4 ([15]). The embedding of the positive cone of $W^{-1,p}(\Omega)$ in $W^{-1,q}(\Omega)$ is completely continuous for $q < p$.

Theorem 4.3 says that compactness in $W_{loc}^{-1,q}$ coupled with boundedness in $W_{loc}^{-1,r}$ yields compactness in $W_{loc}^{-1,p}$. This is useful for our study of the zero relaxation and dissipation limits in Sections 6 and 8. Theorem 4.4 is used in Section 7 for $q = 2$ and $p > 2$.

5. Invariant Regions

A closed set $\Sigma \subset \mathbb{R}^n$ is called an invariant region for solutions $U(x, t)$ of (3.5) if $U(x, t) \in \Sigma, t \geq 0$, whenever the initial data $U(x, 0)$ are in Σ [6]. This notion is useful for studying a priori L^∞ estimates. Consider first the viscous elastic model (2.1):

$$(5.1) \quad \begin{cases} u_t + (f(u) - v)_x = \varepsilon u_{xx}, \\ v_t + \frac{v - (1 - \mu)f(u)}{\delta} = \varepsilon v_{xx}, \end{cases}$$

where $f(0) = 0, f'(u) > 0, f''(0) \gg 1$.

Theorem 5.1. Suppose that there exists $u_* \in (0, \infty)$ with

$$(u - u_*)f''(u) > 0, \quad u \in (0, \infty) - \{u_*\}.$$

Then the following regions are invariant ones for (5.1) for all $\varepsilon > 0$, $\delta > 0$:

$$\Sigma_C = \{(u, v) : 0 \leq v \leq \frac{1-\mu}{\mu}C, \quad f^{-1}(v) \leq u \leq f^{-1}(v+C)\},$$

where

$$C \geq \max\left(\frac{\mu \sup v_0(x)}{1-\mu}, f(u_*)\right).$$

See Fig. 5.1.

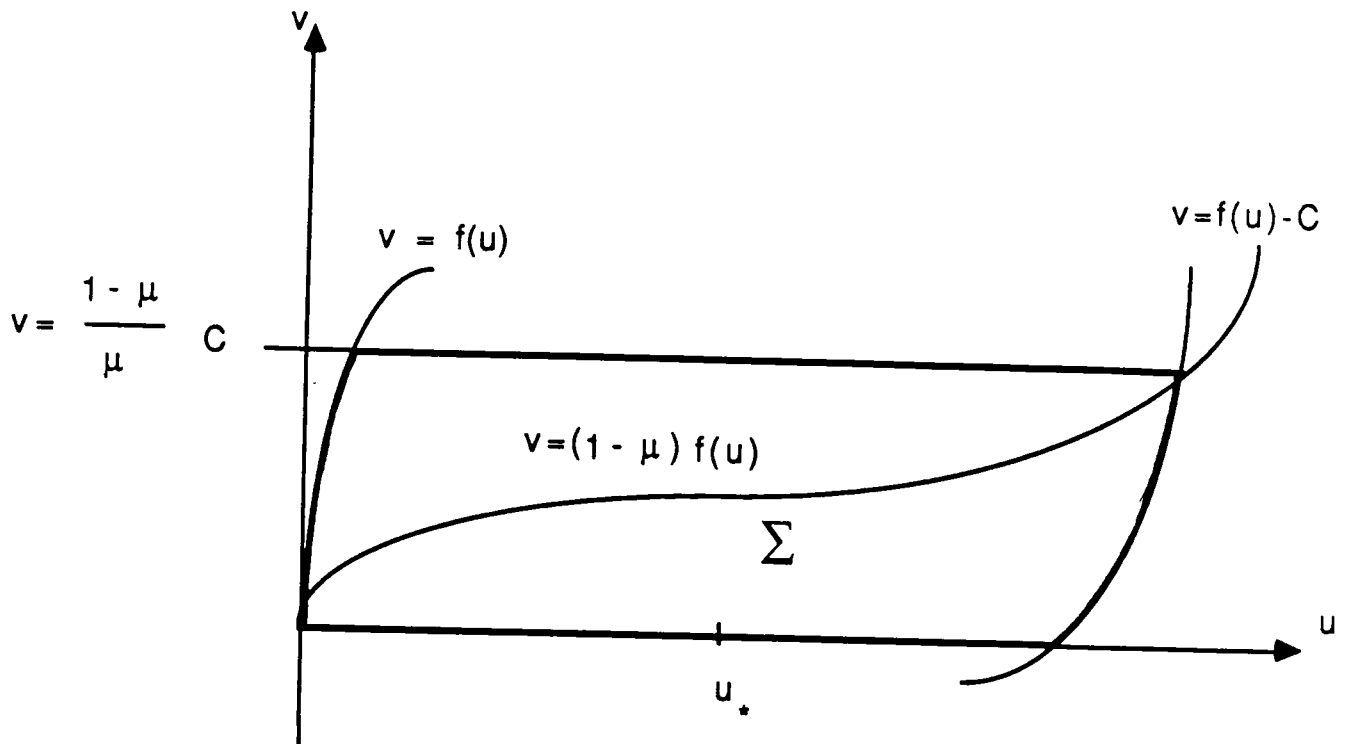


Fig. 5.1

Similarly we have invariant regions for the viscous phase transition model (1.9).

Theorem 5.2. The regions

$$\Sigma_{C_1, C_2} = \{(u, v) : 0 \leq v \leq 1, C_1(1 + (\mu - 1)v) \leq u \leq C_2(1 + (\mu - 1)v)\},$$

with $0 < C_1 < 1 < C_2 < \infty$, are invariant regions for (1.9) for all $\varepsilon, \delta > 0$. See Fig. 5.2.

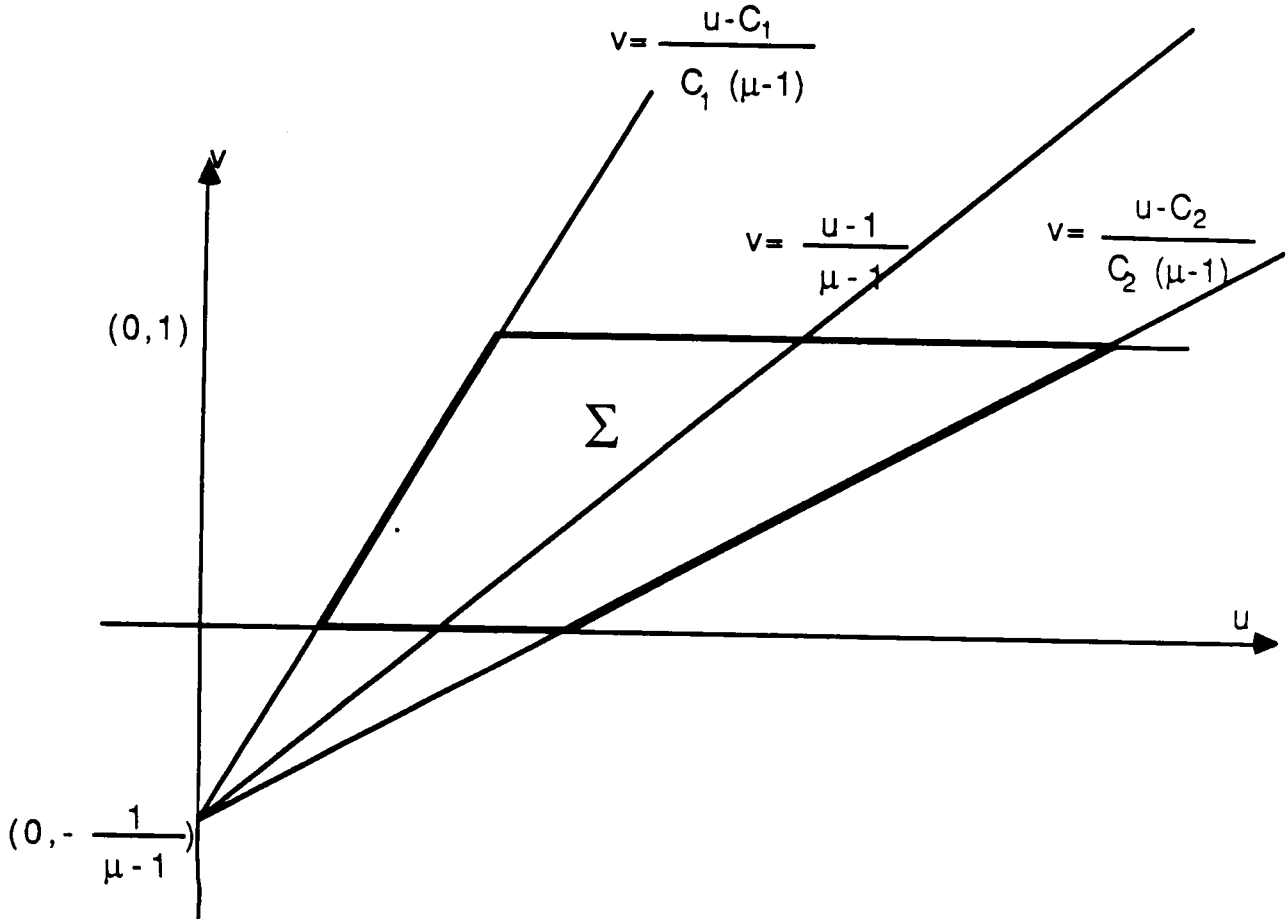


Fig. 5.2

6. Existence of L^∞ Solutions

Consider first the Cauchy problem for the elastic model (2.1):

$$(6.1) \quad \begin{cases} u_t + (f(u) - v)_x = 0, \\ v_t + \frac{v - (1 - \mu)f(u)}{\delta} = 0, \\ (u, v)(x, 0) = (u_0(x), v_0(x)). \end{cases}$$

We shall construct its solution by the zero dissipation limit of solutions of the viscous system

$$(6.2) \quad \begin{cases} u_t + (f(u) - v)_x = \varepsilon u_{xx}, \\ v_t + \frac{v - (1 - \mu)f(u)}{\delta} = \varepsilon v_{xx}, \\ (u, v)(x, 0) = (u_0^\varepsilon(x), v_0^\varepsilon(x)). \end{cases}$$

Theorem 6.1. Suppose that $(u_0(x), v_0(x))$ satisfies

$$(6.3) \quad \begin{cases} 0 \leq v_0(x) \leq f(u_0(x)) \leq M_0 < \infty, \\ (u_0 - \bar{u}, v_0 - \bar{v}) \in L^\infty \cap L^2(-\infty, \infty), \end{cases}$$

for some equilibrium state $(\bar{u}, \bar{v}) = (\bar{u}, (1 - \mu)f(\bar{u}))$. Let $(u_0^\varepsilon, v_0^\varepsilon)$ be some smooth L^2 approximation of (u_0, v_0) . Then (6.2) has global solutions $(u^\varepsilon, v^\varepsilon)(x, t)$ satisfying

$$(6.4) \quad \begin{cases} 0 \leq v^\varepsilon(x, t) \leq f(u^\varepsilon(x, t)) \leq M < \infty, \\ \|v^\varepsilon - (1 - \mu)f(u^\varepsilon)\|_{L^2} \leq \delta M, \\ \sqrt{\varepsilon} \|(u^\varepsilon, v^\varepsilon)_x\|_{L^2} \leq M, \end{cases}$$

for some M independent of ε and δ . Moreover, there exists a sequence $\varepsilon_k \rightarrow 0$ such that

$$(u^{\varepsilon_k}, v^{\varepsilon_k})(x, t) \rightarrow (u, v)(x, t), \quad \text{a.e. ,}$$

and $(u, v)(x, t)$ is a global weak solution for (6.1) that has the same bound as the initial data (6.3),

$$(6.5) \quad 0 \leq v(x, t) \leq f(u(x, t)) \leq M_0,$$

and satisfies the entropy condition

$$(6.6) \quad \eta(u, v)_t + q(u, v)_x + \eta(u, v)_v \frac{v - (1 - \mu)f(u)}{\delta} \leq 0,$$

and the estimate on the deviation from the equilibrium

$$(6.7) \quad \|v - (1 - \mu)f(u)\|_{L^2} \leq \delta M,$$

for some M independent of δ .

Proof: The initial data $(u_0^\varepsilon, v_0^\varepsilon)$ are obtained through the standard convolution with the smooth mollifiers $j_\varepsilon(x)$:

$$(u_0^\varepsilon, v_0^\varepsilon)(x) = \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} (u_0(y), v_0(y)) j_\varepsilon(x-y) dy.$$

The existence of global solutions to (6.2) follows from the usual local existence theory and the a priori sup norm estimate on the solutions by the invariant region argument in Section 5. To show (6.4), we multiply (6.2) by $\nabla \bar{\eta}_*(u^\varepsilon, v^\varepsilon)$ with

$$\bar{\eta}_*(u, v) \equiv \bar{\eta}_*(u, v) - \bar{\eta}_*(\bar{u}, \bar{v}) - \nabla \bar{\eta}_*(\bar{u}, \bar{v}) \begin{pmatrix} u - \bar{u} \\ v - \bar{v} \end{pmatrix},$$

and obtain

$$(6.8) \quad \begin{aligned} & \bar{\eta}_*(u^\varepsilon, v^\varepsilon)_t + \bar{q}_*(u^\varepsilon, v^\varepsilon)_x \\ &= \bar{\eta}_*(u^\varepsilon, v^\varepsilon)_v \frac{(1-\mu)f(u^\varepsilon) - v^\varepsilon}{\delta} + \varepsilon \bar{\eta}_*(u^\varepsilon, v^\varepsilon)_{xx} - \varepsilon (u^\varepsilon, v^\varepsilon)_x \nabla^2 \eta_*(u^\varepsilon, v^\varepsilon) \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}_x. \end{aligned}$$

The estimate (6.4) follows from integrating the above equality over $(-\infty, \infty) \times (0, t)$ by using (3.7). Similarly, (6.6) and (6.7) follow from (6.4) and the above equality in the limit $\varepsilon \rightarrow 0$.

It remains to show the convergence of $(u^\varepsilon, v^\varepsilon)$ as $\varepsilon \rightarrow 0$. To apply the theory of compensated compactness in Section 4, we need to check the condition (4.2). Let (η, q) be any entropy pair. Multiplying (6.2) by $\nabla \eta$, we have

$$\eta(u^\varepsilon, v^\varepsilon)_t + q(u^\varepsilon, v^\varepsilon)_x = \sum_{i=1}^3 I_i^\varepsilon,$$

where

$$\begin{cases} I_1^\varepsilon = \eta(u^\varepsilon, v^\varepsilon)_v \frac{(1-\mu)f(u^\varepsilon) - v^\varepsilon}{\delta}, \\ I_2^\varepsilon = -\varepsilon (u^\varepsilon, v^\varepsilon)_x \nabla^2 \eta(u^\varepsilon, v^\varepsilon) \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}_x, \\ I_3^\varepsilon = \varepsilon \eta(u^\varepsilon, v^\varepsilon)_{xx}. \end{cases}$$

From (6.4) we have

$$\int_0^T \int_{-\infty}^{\infty} (|I_1^\varepsilon| + |I_2^\varepsilon|) dx dt \leq M,$$

and, therefore, $I_1^\varepsilon + I_2^\varepsilon$ is compact in $W_{loc}^{-1,p}$, $1 < p < 2$. Moreover, from the second estimate in (6.4),

$$\|I_2^\varepsilon\|_{H_{loc}^{-1}} = O(1)\sqrt{\varepsilon} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

we have I_2^ε is compact in H_{loc}^{-1} . Since $(u^\varepsilon, v^\varepsilon)$ is uniformly bounded, we obtain that

$$\eta(u^\varepsilon, v^\varepsilon)_t + q(u^\varepsilon, v^\varepsilon)_x$$

are compact in $W_{loc}^{-1,p}(\Omega)$, $1 < p < 2$, and bounded in $W_{loc}^{-1,r}(\Omega)$, $r > 2$. By Theorem 4.3, we conclude that

$$\eta(u^\varepsilon, v^\varepsilon)_t + q(u^\varepsilon, v^\varepsilon)_x \quad \text{compact in } H_{loc}^{-1},$$

and (4.2) follows. Thus we have from Theorem 4.2 that

$$u^\varepsilon(x, t) \rightarrow u(x, t) \quad \text{a.e. as } \varepsilon \rightarrow 0.$$

The convergence of v^ε is a consequence of the convergence of u^ε . From the second equation in (6.2) we have

$$\begin{aligned} v^\varepsilon(x, t) = & e^{-\frac{t}{\delta}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4\epsilon t}} v_0^\varepsilon(y) dy \\ & + \frac{1-\mu}{\delta} \int_0^t \int_{-\infty}^{\infty} e^{-\frac{t-\tau}{\delta}} f(u^\varepsilon(y, \tau)) \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4\epsilon(t-\tau)}} dy d\tau. \end{aligned}$$

It is clear from the above equality that, for Lipschitz function f , the convergence of u^ε implies that

$$v^\varepsilon \rightarrow v \quad \text{a.e. as } \varepsilon \rightarrow 0,$$

for some L^∞ function v satisfying

$$(6.9) \quad v(x, y) = e^{-\frac{t}{\delta}} v_0(x, t) + \frac{1-\mu}{\delta} \int_0^t e^{-\frac{t-\tau}{\delta}} f(u(x, \tau)) d\tau.$$

It is clear that (u, v) is a weak solution of (6.1).

By the same approach we have the existence theory for the phase transition model (1.9):

$$(6.10) \quad \begin{cases} u_t + \left(\frac{1 + (\mu - 1)v}{u} \right)_x^2 = 0, \\ v_t = \frac{V_*(u) - v}{\delta}, \quad \mu > 1, \delta > 0, \\ (u, v)(x, 0) = (u_0(x), v_0(x)), \end{cases}$$

from

$$(6.11) \quad \begin{cases} u_t + \left(\frac{1 + (\mu - 1)v}{u} \right)_x^2 = \varepsilon u_{xx}, \\ v_t = \frac{V_*(u) - v}{\delta} + \varepsilon v_{xx}, \\ (u, v)(x, 0) = (u_0^\varepsilon(x), v_0^\varepsilon(x)), \end{cases}$$

where

$$(6.12) \quad V_*(u) = \begin{cases} 0, & 0 < u < 1, \\ \frac{u-1}{\mu-1}, & 1 < u < \mu, \\ 1, & u > \mu. \end{cases}$$

Theorem 6.2. Suppose that $(u_0(x) - \bar{u}, v_0(x) - \bar{v}) \in L^2 \cap L^\infty(-\infty, \infty)$ for some equilibrium state $(\bar{u}, \bar{v}) = (\bar{u}, V_*(\bar{u}))$, and that, for some $C_1 > 0$ and $C_2 > 0$,

$$0 \leq v_0(x) \leq 1, \quad C_1(1 + (\mu - 1)v_0(x)) \leq u_0(x) \leq C_2(1 + (\mu - 1)v_0(x)).$$

Then there exist $(u_0^\varepsilon(x), v_0^\varepsilon(x))$ and the corresponding solutions $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ of (6.11) such that for some $\varepsilon_k \rightarrow 0$,

$$(u^{\varepsilon_k}(x, t), v^{\varepsilon_k}(x, t)) \rightarrow (u(x, t), v(x, t)), \quad \text{a.e.}$$

Moreover, $(u(x, t), v(x, t))$ is a weak solution of (6.10) satisfying

$$(6.13) \quad 0 \leq v(x, t) \leq 1, \quad C_1(1 + (\mu - 1)v(x, t)) \leq u(x, t) \leq C_2(1 + (\mu - 1)v(x, t)),$$

$$(6.14) \quad \eta(u, v)_t + q(u, v)_x + \nabla \eta(u, v) \frac{v - V_*(u)}{\delta} \leq 0,$$

for any C^2 convex entropy pair (η, q) .

7. Zero Relaxation Limit for the Elastic Model

We consider the limit of the solutions (u, v) of (6.1) in Theorem 6.1 as the relaxation time $\delta \rightarrow 0$. To indicate the dependence of (u, v) on δ , we write (u, v) of Theorem 6.1 as (u_δ, v_δ) .

Theorem 7.1. The solutions $(u_\delta, v_\delta)(x, t)$ of (6.1) with initial data satisfying (6.3) converge to bounded measurable functions $(u, v)(x, t)$ a.e. in $t > 0$ as $\delta \rightarrow 0$. Moreover, $u = u(x, t)$ is an admissible weak solution of the Cauchy problem of the equilibrium equation:

$$(7.1) \quad \begin{cases} u_t + (\mu f(u))_x = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

and $v(x, t)$ equals the equilibrium value $(1 - \mu)f(u(x, t))$ for any $t > 0$.

Proof: For convergence and admissibility, we need to estimate the entropy inequality for (7.1) based on the entropy condition (6.6):

$$(7.2) \quad \eta(u_\delta, v_\delta)_t + q(u_\delta, v_\delta)_x + \eta_v(u_\delta, v_\delta) \frac{v_\delta - V_*(u_\delta)}{\delta} \leq 0.$$

We first claim that, given any entropy pair (η, q) for (6.1) with $\eta_v(u, v)$ vanishing on the equilibrium curve $v = V_*(u) \equiv (1 - \mu)f(u)$, the pair

$$(7.3) \quad \begin{cases} \eta_*(u) = \eta(u, (1 - \mu)f(u)), \\ q_*(u) = q(u, (1 - \mu)f(u)), \end{cases}$$

forms an entropy pair for (7.1), that is,

$$q'_*(u) = \mu\eta'_*(u)f'(u).$$

Conversely, given any entropy pair $(\eta_*(u), q_*(u))$ for (7.1), there exists an entropy pair $(\eta(u, v), q(u, v))$ for (6.1) such that $\eta_v(u, (1 - \mu)f(u)) \equiv 0$ and (7.3) hold. The first half of the claim follows from the compatibility condition (3.2) for (η, q) and is valid for general relaxation models. The condition that η_v vanishes on the equilibrium curve $v = V_*(u)$ is needed, of course. To verify the second half of the claim, we notice from the general expression (3.5) for (η, q) that, for any given function $\eta_*(u)$, we need only to find a function H such that

$$\int^u H(\mu f(\xi))d\xi = \eta_*(u).$$

Differentiate the above equality, and set $w = \mu f(u)$ to obtain

$$H(w) = \eta'_* \left(f^{-1} \left(\frac{w}{\mu} \right) \right).$$

Next, note from (6.7) that

$$\begin{aligned} & \eta_*(u_\delta)_t + q_*(u_\delta)_x \\ &= \eta(u_\delta, (1 - \mu)f(u_\delta))_t + q(u_\delta, (1 - \mu)f(u_\delta))_x \\ &= \eta(u_\delta, v_\delta)_t + q(u_\delta, v_\delta)_x + \Delta_\delta, \end{aligned}$$

where

$$\begin{aligned} & \|\Delta_\delta\|_{H_{loc}^{-1}(\Omega)} \\ & \leq C \int_0^T \int_\Omega \left(\left(\eta(u_\delta, v_\delta) - \eta(u_\delta, (1 - \mu)f(u_\delta)) \right)^2 + \left(q(u_\delta, v_\delta) - q(u_\delta, (1 - \mu)f(u_\delta)) \right)^2 \right) dxdt \\ & \leq C \int_0^T \int_\Omega (v_\delta - (1 - \mu)f(u_\delta))^2 dxdt \\ & \leq C\delta \rightarrow 0, \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

Thus the convergence of u_δ to u follows from the theory of compensated compactness as in the proof of Theorem 6.1. The admissibility of u follows from the above estimate and (6.6). Finally, the convergence of $v_\delta(x, t)$, $t > 0$, to its equilibrium value $(1 - \mu)f(u(x, t))$ follows from (6.9) and the convergence of u_δ to u :

$$(7.4) \quad v_\delta(x, t) = e^{-\frac{t}{\delta}} v_0(x, t) + \frac{1 - \mu}{\delta} \int_0^t e^{-\frac{t-\tau}{\delta}} f(u_\delta(x, \tau)) d\tau.$$

This completes the proof of the theorem.

We remark from (7.4) that the convergence of $v_\delta(x, t)$ to $v(x, t)$ is singular as $t \rightarrow 0$. In other words, an initial layer develops as δ becomes small. This is expected as we do not assume the initial data $(u_0(x), v_0(x))$ to be in the equilibrium.

8. Zero Relaxation and Dissipation Limit for the Phase Transition Model

Consider the viscous phase transition model (6.11):

$$(8.1) \quad \begin{cases} u_t + \left(\frac{1 + (\mu - 1)v}{u} \right)_x = \varepsilon u_{xx}, \\ v_t = \frac{V_*(u) - v}{\delta} + \varepsilon v_{xx}, \\ (u, v)(x, 0) = (u_0^\varepsilon(x), v_0^\varepsilon(x)), \end{cases}$$

where

$$V_*(u) = \begin{cases} 0, & 0 < u < 1, \\ \frac{u-1}{\mu-1}, & 1 < u < \mu, \\ 1, & u > \mu. \end{cases}$$

We wish to show that the solution $(u, v)(x, t) \equiv (u_\delta^\varepsilon, v_\delta^\varepsilon)(x, t)$ converges to that of the inviscid equilibrium equations

$$(8.2) \quad \begin{cases} u_t + \left(\frac{1 + (\mu - 1)V_*(u)}{u} \right)_x = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

as the relaxation time δ and the viscosity ε tend to zero. Moreover, for stability, we require δ to tend to zero faster than ε . We assume that the initial data are bounded:

$$(8.3) \quad \begin{cases} \|u_0(x) - \bar{u}, v_0(x) - \bar{v}\|_{L^2} \leq M_0, \\ 0 \leq v_0(x) \leq 1, \\ C_1(1 + (\mu - 1)v_0(x)) \leq u_0(x) \leq C_2(1 + (\mu - 1)v_0(x)), \end{cases}$$

for some equilibrium state (\bar{u}, \bar{v}) and positive constants M_0, C_1 , and C_2 .

Lemma 8.1. The solution (u, v) of (8.1) satisfies

$$(8.4) \quad \begin{cases} 0 \leq v(x, t) \leq 1, \\ C_1(1 + (\mu - 1)v(x, t)) \leq u(x, t) \leq C_2(1 + (\mu - 1)v(x, t)), \end{cases}$$

$$(8.5) \quad \int_0^T \int_{-\infty}^{\infty} (V_*(u) - v)^2 \leq \delta M,$$

$$(8.6) \quad \sqrt{\delta}\varepsilon \int_0^T \int_{-N}^N (u_x^2 + v_x^2 + u_t^2) dx dt \leq M,$$

for some M depending on T and N but not on ε and δ .

Proof: The estimates (8.4) and (8.5) follow from (6.13), (6.14), and (3.13). In fact, the entropy inequality, by multiplying (8.1) by $\nabla\eta_*$ with

$$(8.7) \quad \bar{\eta}_* \equiv \bar{\eta}_*(u, v) - \bar{\eta}_*(\bar{u}, \bar{v}) - \nabla\bar{\eta}_*(\bar{u}, \bar{v}) \begin{pmatrix} u - \bar{u} \\ v - \bar{v} \end{pmatrix},$$

and then integrating, yields (8.5) and

$$(8.8) \quad \varepsilon \int_0^T \int_{-\infty}^{\infty} \left(u_x + \frac{(\mu-1)u}{1+(\mu-1)v} v_x \right)^2 dx dt \leq M.$$

Because of the degeneracy of (8.1), $\nabla^2\bar{\eta}_*$ is not strictly convex and, therefore, (8.8) does not yield effective estimate for u_x and v_x . To obtain (8.6), we multiply the second equation of (8.1) by $2\sqrt{\delta}\varphi^2v$, where φ is a smooth function with

$$(8.9) \quad \varphi(x) \equiv \begin{cases} 1, & x \in [-N, N], \\ 0, & x \notin [-N-\beta, N+\beta], \beta > 0, \end{cases}$$

to get

$$\sqrt{\delta}(\varphi^2v^2)_t \leq 2\varphi^2v \frac{V_*(u) - v}{\sqrt{\delta}} + 2\sqrt{\delta}\varepsilon(\varphi^2vv_x)_x - \sqrt{\delta}\varepsilon\varphi^2v_x^2 + C\sqrt{\delta}\varepsilon\varphi_x^2v^2.$$

Thus the boundedness of (u, v) , (8.4), (8.5), and the above equality yield

$$(8.10) \quad \sqrt{\delta}\varepsilon \int_0^T \int_{-N}^N v_x^2 dx dt \leq M \left(1 + \int_0^T \int_{-\infty}^{\infty} \frac{(V_*(u) - v)^2}{\delta} dx dt \right) \leq M.$$

From (8.10) and (8.8) we have

$$(8.11) \quad \sqrt{\delta}\varepsilon \int_0^T \int_{-N}^N u_x^2 dt dx \leq M.$$

Similarly, multiplying the first equation of (8.1) by φ^2u_{xx} and integrating, we obtain

$$(8.12) \quad \begin{aligned} & \varepsilon \int_0^T \int_{-\infty}^{\infty} u_{xx}^2 \varphi^2 dx dt + \int_{-\infty}^{\infty} u_x^2 \varphi(x, T)^2 dx \\ & \leq C \left[\int_{-\infty}^{\infty} u_{0x}^2 \varphi^2 dx + \frac{1}{\varepsilon} \int_0^T \int_{-\infty}^{\infty} (u_x^2 + v_x^2) \varphi^2 dx dt + \varepsilon \int_0^T \int_{-\infty}^{\infty} u_t^2 \varphi^2 dx dt \right]. \end{aligned}$$

We have from the first equation of (8.1)

$$(8.13) \quad u_t^2 \leq 2 \left(\varepsilon^2 u_{xx}^2 + \left[2 \left(\frac{1 + (\mu - 1)v}{u} \right) \left(\frac{1 + (\mu - 1)v}{u} \right)_x \right]^2 \right).$$

Finally we have from (8.10)–(8.13) that

$$\sqrt{\delta} \varepsilon^3 \int_0^T \int_{-N}^N u_{xx}^2 dx dt + \sqrt{\delta} \varepsilon \int_0^T \int_{-N}^N u_t^2 dx dt \leq M.$$

This completes the proof of Lemma 8.1.

To relate the solutions $(u, v) \equiv (u_\delta^\varepsilon, v_\delta^\varepsilon)$ of (8.1) to those of (8.2), we note from the second equation of (8.1) that

$$(8.14) \quad \begin{aligned} v(x, t) = & e^{-\frac{t}{\delta}} \int_{-\infty}^{\infty} G^\varepsilon(x - y, t) v_0(y) dy \\ & + \frac{1}{\delta} \int_{-\infty}^{\infty} \int_0^t G^\varepsilon(x - y, t - \tau) e^{-\frac{t-\tau}{\delta}} V_*(u(y, \tau)) dy d\tau, \end{aligned}$$

where

$$G^\varepsilon(x, t) \equiv \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{x^2}{4\varepsilon t}}.$$

Substituting this into the first equation of (8.11), we have, for some $0 \leq \theta = \theta(x, t) \leq 1$,

$$(8.15) \quad \left\{ \begin{aligned} & u_t + f(u, V_*(u))_x \\ & = \varepsilon u_{xx} - \left(f_v(u, V_*(u) + \theta(v - V_*(u))) \cdot \left(e^{-t/\delta} \int_{-\infty}^{\infty} G^\varepsilon(x - y, t) v_0(y) dy \right. \right. \\ & \quad \left. \left. + \int_{-\infty}^{\infty} \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x - y, t - \tau) (V_*(u(y, \tau)) - V_*(u(x, t))) dy d\tau \right) \right)_x, \end{aligned} \right.$$

with

$$f(u, v) \equiv \left(\frac{1 + (\mu - 1)v}{u} \right)^2.$$

This measures the deviation from (8.2) for the solutions of (8.1). We also need to study entropy pairs (E, F) with $F' = E' f'$ for (8.2). Multiply (8.15) by $E'(u)$ to yield

$$(8.16) \quad E(u)_t + F(u)_x = \sum_{i=1}^6 I_i,$$

where

$$\left\{ \begin{array}{l} I_1 \equiv \varepsilon E(u)_{xx} , \\ I_2 \equiv -\varepsilon E''(u)u_x^2 , \\ I_3 \equiv -\left(E'(u)f_v(u, V_*(u) + \theta(v - V_*(u)))e^{-\frac{t}{\delta}} \int_{-\infty}^{\infty} G^\varepsilon(x-y, t)v_0(y)dy \right)_x , \\ I_4 \equiv E''(u)f_v(u, V_*(u) + \theta(v - V_*(u)))u_x e^{-\frac{t}{\delta}} \int_{-\infty}^{\infty} G^\varepsilon(x-y, t)v_0(y)dy , \\ I_5 \equiv -\left(E'(u)f_v(u, V_*(u) + \theta(v - V_*(u))) \right. \\ \quad \times \left. \int_{-\infty}^{\infty} \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) (V_*(u(y, \tau)) - V_*(u(x, t))) d\tau dy \right)_x , \\ I_6 \equiv E''(u)f_v(u, V_*(u) + \theta(v - V_*(u)))u_x \\ \quad \times \int_{-\infty}^{\infty} \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) (V_*(u(y, \tau)) - V_*(u(x, t))) d\tau dy . \end{array} \right.$$

Lemma 8.2. For some $M > 0$ independent of ε and δ , we have

$$\sqrt{\varepsilon}(\|u_x\|_{L^2_{loc}} + \|v_x\|_{L^2_{loc}} + \|u_t\|_{L^2_{loc}}) \leq M,$$

for $\frac{\delta}{\varepsilon} \ll 1$.

Proof. Multiply (8.15) by $\varphi(x)$, use (8.9), and integrate to yield (8.17)

$$\begin{aligned} & \varepsilon \int_0^\infty \int_{-\infty}^\infty u_x^2 \varphi^2 dx dt \\ & \leq C \left[\int_{-\infty}^\infty (u_0 - \bar{u})^2 \varphi^2 dx + \int_0^\infty \int_{-\infty}^\infty |\varphi \varphi_x| \int^u \xi f'(\xi) d\xi dx dt \right. \\ & \quad + \varepsilon \int_0^\infty \int_{-\infty}^\infty |\varphi u \varphi_x u_x| dx dt + \int_0^\infty \int_{-\infty}^\infty |\varphi \varphi_x| e^{-\frac{t}{\delta}} dx dt \int_{-\infty}^\infty G^\varepsilon(x-y, t) |v_0(y)| dy \\ & \quad + \int_0^\infty \int_{-\infty}^\infty \varphi^2 |u_x| e^{-\frac{t}{\delta}} dx dt \int_{-\infty}^\infty G^\varepsilon(x-y, t) |v_0(y)| dy \\ & \quad + \int_0^\infty \int_{-\infty}^\infty |\varphi \varphi_x| dx dt \int_{-\infty}^\infty \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) |V_*(u(y, \tau)) - V_*(u(x, t))| d\tau dy \\ & \quad \left. + \int_0^\infty \int_{-\infty}^\infty \varphi^2 u_x^2 dx dt \int_{-\infty}^\infty \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) |V_*(u(y, \tau)) - V_*(u(x, t))| d\tau dy \right]. \end{aligned}$$

Noting that $\varphi(x)$ has compact support (8.9), we have

$$(8.18) \quad \varepsilon \int_0^\infty \int_{-\infty}^\infty |\varphi \varphi_x u u_x| dx dt \leq \frac{\varepsilon}{L} \int_0^\infty \int_{-\infty}^\infty u_x^2 \varphi^2 dx dt + CL\varepsilon ,$$

and

$$(8.19) \quad \int_0^\infty \int_{-\infty}^\infty \varphi^2 |u_x| e^{-\frac{t}{\delta}} dx dt \int_{-\infty}^\infty G(x-y, t) |v_0(y)| dy \leq \frac{\varepsilon}{L} \int_0^\infty \int_{-\infty}^\infty |u_x \varphi|^2 dx dt + CL \frac{\delta}{\varepsilon} ,$$

for sufficiently large $L > 0$.

From Lemma 8.1, we obtain

$$(8.20) \int_0^\infty \int_{-\infty}^\infty |\varphi \varphi_x| dx dt \int_{-\infty}^\infty \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) \left| V_*(u(y, \tau)) - V_*(u(x, t)) \right| d\tau dy \\ \leq \frac{\varepsilon}{L} \int_0^\infty \int_{-\infty}^\infty \varphi^2 u_x^2 dx dt + CL\delta^{\frac{1}{4}},$$

and

$$(8.21) \int_0^\infty \int_{-\infty}^\infty \varphi^2 |u_x| dx dt \int_{-\infty}^\infty \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) \left| V_*(u(y, \tau)) - V_*(u(x, t)) \right| d\tau dy \\ \leq \frac{\varepsilon}{L} \int_0^\infty \int_{-\infty}^\infty \varphi^2 u_x^2 dx dt + CL,$$

for sufficiently large $L > 0$.

From (8.17)–(8.21), we have

$$\varepsilon \int_0^\infty \int_{-\infty}^\infty (u_x \varphi)^2 dx dt \leq \varepsilon \frac{4C}{L} \int_0^\infty \int_{-\infty}^\infty (u_x \varphi)^2 dx dt + CL.$$

Set $L = 8C$. We then obtain

$$(8.22) \quad \varepsilon \int_0^T \int_{-N}^N u_x^2 dx dt \leq M.$$

This fact and (8.8) yield

$$(8.23) \quad \varepsilon \int_0^T \int_{-N}^N v_x^2 dx dt \leq M.$$

By a similar process in deriving (8.14), we have from (8.22) and (8.23) that

$$\varepsilon^3 \int_0^T \int_{-N}^N u_{xx}^2 dx dt + \varepsilon \int_0^T \int_{-N}^N u_t^2 dx dt \leq M.$$

This completes the proof of Lemma 8.2.

Lemma 8.3. For $\delta/\varepsilon \ll 1$, we have, for C^2 entropy pairs $(E(u), F(u))$ with $F' = E' f'$ and solutions (u, v) of (8.1),

$$E(u)_t + F(u)_x \quad \text{compact in } H_{loc}^{-1}.$$

Proof. Since (u, v) is bounded from (8.4), we have

$$E(u)_t + F(u)_x \quad \text{bounded in } W_{loc}^{-1, \infty}.$$

Thus by Theorem 4.3, it suffices from (8.16) to prove

$$(8.24) \quad \sum_{i=1}^6 I_i \quad \text{compact in } W_{loc}^{-1,p}, \quad 1 < p < 2 .$$

From Lemmas 8.1 and 8.2, we easily deduce

$$(8.25) \quad I_1 + I_2 \quad \text{compact in } W_{loc}^{-1,p}, \quad 1 < p < 2,$$

by direct calculations. Moreover,

$$\begin{aligned} \|I_3\|_{H_{loc}^{-1}} &= \sup_{\|\phi\|_{H_0^1} \leq 1} \left| \int \int I_3 \phi dx dt \right| \\ &\leq C \sup_{\|\phi\|_{H_0^1} \leq 1} \left| \int \int e^{-\frac{t}{\delta}} |\phi_x| dx dt \right| \\ &\leq C \left[\int_{-N}^N \int_0^T e^{-\frac{2t}{\delta}} dx dt \right]^{1/2} \\ &\leq C \sqrt{\delta} \rightarrow 0, \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

and, therefore,

$$(8.26) \quad I_3 \quad \text{compact in } H_{loc}^{-1} .$$

Similarly we have

$$\begin{aligned} &\int_0^T \int_{-N}^N |I_4| dx dt \\ &\leq C \int_{-N}^N dx \int_0^T |u_x| e^{-\frac{t}{\delta}} \left| \int_{-\infty}^{\infty} G^\varepsilon(x-y, t) v_0(y) dy \right| dt \\ &\leq \int_0^T \int_{-N}^N |u_x| e^{-\frac{t}{\delta}} dx dt \\ &\leq C \left(\int_0^T \int_{-N}^N u_x^2 dx dt \right)^{1/2} \left(\int_0^T \int_{-N}^N e^{-\frac{2t}{\delta}} dx dt \right)^{1/2} \\ &\leq C \left(\frac{\delta}{\varepsilon} \right)^{1/2}, \quad \text{for any } N > 0, \end{aligned}$$

whence

$$(8.27) \quad I_4 \quad \text{compact in } W_{loc}^{-1,p}, \quad 1 < p < 2 .$$

Finally, we have

$$\begin{aligned}
& \|I_5\|_{H_{loc}^{-1}} \\
& \leq C \sup_{\|\phi\|_{H_0^1} \leq 1} \int_0^\infty \int_{-\infty}^\infty |\phi_x| \left| \int_{-\infty}^\infty \int_0^t \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) (V_*(u(y, \tau)) - V_*(u(x, t))) d\tau dy \right| dx dt \\
& \leq C \left(\int_0^T \int_{-N}^N \left(\int_0^t \int_{-\infty}^\infty \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) (V_*(u(y, \tau)) - V_*(u(x, t))) d\tau dy \right)^2 dx dt \right)^{\frac{1}{2}} \\
& \leq C \left(\varepsilon \delta \|u_x\|_{L_{loc}^2}^2 + \delta^2 \|u_t\|_{L_{loc}^2}^2 \right)^{\frac{1}{2}} \\
& \leq C \sqrt{\delta} \rightarrow 0, \quad \text{as } \delta \rightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^T \int_{-N}^N |I_6| dx dt \\
& \leq C \int_0^T \int_{-N}^N |u_x| \int_0^t \int_{-\infty}^\infty \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) |V_*(u(y, \tau)) - V_*(u(x, t))| dy d\tau dx dt \\
& \leq C \left(\int_0^T \int_{-N}^N u_x^2 dx dt \right)^{\frac{1}{2}} \\
& \quad \times \left(\int_0^T \int_{-N}^N \left(\int_0^t \int_{-\infty}^\infty \frac{1}{\delta} e^{-\frac{t-\tau}{\delta}} G^\varepsilon(x-y, t-\tau) |V_*(u(y, \tau)) - V_*(u(x, t))| dy d\tau \right)^2 dx dt \right)^{\frac{1}{2}} \\
& \leq C \varepsilon^{-\frac{1}{2}} \left(\varepsilon \delta \|u_x\|_{L_{loc}^2}^2 + \delta^2 \|u_t\|_{L_{loc}^2}^2 \right)^{\frac{1}{2}} \\
& \leq C \sqrt{\frac{\delta}{\varepsilon}} \leq C.
\end{aligned}$$

Thus we have

$$(8.28) \quad I_5, \quad I_6 \quad \text{compact in } W_{loc}^{-1,p}, \quad 1 < p < 2.$$

The lemma follows from (8.24)–(8.28).

Theorem 8.1. The solutions $(u, v) = (u_\delta^\varepsilon, v_\delta^\varepsilon)$ of (8.1) in Theorem 6.2 converge strongly to an L^∞ function (\bar{u}, \bar{v}) a.e. as ε and δ/ε tend to zero. Moreover, \bar{u} is the unique admissible weak solution of (8.2), and

$$(8.29) \quad \bar{v}(x, t) = V_*(\bar{u}(x, t)), \quad \text{a.e. for } t > 0.$$

With Lemmas 8.1 and 8.2, the proof of Theorem 8.1 follows from the same argument as in the proof of Theorem 7.1. As with the elastic model, $v(x, t)$ also has an initial layer at $t = 0$ and (8.29) does not hold at $t = 0$.

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