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ZEROS OF BESSEL FUNCTIONS: MONOTONICITY, CONCAVITY, INEQUALITIES

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We present a survey of the most important inequalities and monotonicity, concavity (convexity) results of the zeros of Bessel functions. The results refer to the definition $j_{\nu\kappa}$ of the zeros of $C_\nu(x) = J_\nu(x)\ \cos\alpha - Y_\nu(x)\ \sin\alpha$, formulated in [6], where κ is a continuous variable. Sometimes, also the Sturm comparison theorem is an important tool of our results.

1. Introduction

We let $c_{\nu k}$ denote the kth positive x-zero of a cylinder function

$$C_\nu(x) = C_\nu(x,\alpha) = J_\nu(x)\ \cos\alpha - Y_\nu(x)\ \sin\alpha, \qquad 0 \leq \alpha < \pi, \tag{1}$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively. As usual we omit the α-dependence in this notation. When $\alpha = 0$ we use $j_{\nu k}$ for the kth positive x-zero of $J_\nu(x)$.

We can discuss the variation of the positive zeros of $C_\nu(x,\alpha)$ with respect to any of the three variables ν, α, k (the rank of a zero). However, α and k are not really independent, but they can be subsumed in a single variable $\kappa = k - \alpha/\pi$.

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To see this, we observe that for $\nu \geq 0$ the zeros of $C_\nu(x,\alpha)$, $0 < \alpha < \pi$, are the roots of the equation

$$\frac{Y_\nu(x)}{J_\nu(x)} = \cot\alpha, \qquad 0 < \alpha < \pi. \tag{2}$$

The graph of the functions of the left-hand side consists of branches which increase from $-\infty$ to $+\infty$ in each interval $(0, j_{\nu 1})$, $(j_{\nu k}, j_{\nu,k+1})$, $k = 1,2,\ldots$, between the positive zeros of $J_\nu(x)$. This can be seen by the relation

$$\frac{d}{dx}\frac{Y_\nu(x)}{J_\nu(x)} = \frac{J_\nu(x)Y_\nu'(x) - J_\nu'(x)Y_\nu(x)}{J_\nu^2(x)} = \frac{2}{\pi x J_\nu^2(x)},$$

where the last equality follows from the Wronskian relation [47, p. 76]. As α decreases from π to 0, $\cot\alpha$ increases from $-\infty$ to $+\infty$, so each zero of $C_\nu(x,\alpha)$ increases from one positive zero $j_{\nu k}$ of $J_\nu(x)$ to the next larger one $j_{\nu,k+1}$. This shows that it makes sense to define $j_{\nu\kappa}$ for any real $\kappa \geq 0$ by $j_{\nu 0} = 0$, $j_{\nu\kappa} = c_{\nu k}(\alpha)$ when k is the largest integer less than $\kappa + 1$ and $\alpha = \pi(k-\kappa)$. This shows also that $j_{\nu\kappa}$ is a continuous increasing function of κ on $[0,\infty)$. The positive zeros of $J_\nu(x)$ correspond to positive integer values of κ, i.e. for $\kappa = 1,2,\ldots$ we get $j_{\nu k}$. Similarly, $j_{\nu,k-1/2} = y_{\nu k}$, $k = 1,2,\ldots$ where $y_{\nu k}$ denotes the kth zero of $Y_\nu(x)$. In [6] it was proved that $j_{\nu\kappa}$ is the unique solution of the integro-differential equation

$$\frac{d}{d\nu}\, j = 2j \int_0^\infty K_0\,(2j\sinh t)\, e^{-2\nu t} dt \tag{3}$$

which satisfy $j(\nu) \to 0$, as $\nu \to -\kappa^+$ (here $K_0(x)$ denotes the modified Bessel function). This is motivated by the formula [47, p. 408]

$$\frac{d}{d\nu}\, c_{\nu k} = 2c_{\nu k} \int_0^\infty K_0\,(2c_{\nu k}\sinh t)\, e^{-2\nu t} dt \tag{4}$$

and the fact that, for $\nu > 0$ $c_{\nu k}$ may be extended in a continuous way to $\nu < 0$ and $c_{\nu k} \to 0$ as $\nu \to -(k - \alpha/\pi)$. By (3) we have also that $j_{\nu\kappa}$ is an infinitely differentiable function of κ.

This notation for the zeros of the cylinder functions is very useful in the investigations of concavity (convexity) properties of $j_{\nu\kappa}$, essentially with respect to ν, but occasionally, with respect to κ. The study of concavity (convexity) properties are motivated by the fact that they are useful because arise in the quantum mechanical explanation for the origin of the vortex lines produced in superfluid helium when its container is rotated. This explanation has been proposed by Putterman, Kac and Uhlenbeck [42].

The proofs are based on the Sturm comparison theorem and on the Watson formula (4).

2. The Sturm comparison theorem

There are several formulations of the Sturm comparison theorem. For our purposes the following form is the most useful [45, p. 19].

Lemma 2.1. *Let the functions y and Y be nontrivial solutions of the differential equations*

$$y'' + f(x)\, y = 0\,, \qquad Y'' + F(x)\, Y = 0 \tag{5}$$

and let them have consecutive zeros at $x_1, x_2, \dots, x_m$ and $X_1, X_2, \dots, X_m$, respectively, on an interval (a,b). Suppose that f and F are continuous on (a,b), that

$$f(x) < F(x)\,, \qquad 0 < x < x_m \tag{6}$$

and that

$$\lim_{x \to a^+} \left[y'(x)\, Y(x) - y(x)\, Y'(x) \right] = 0. \tag{7}$$

Then

$$X_k < x_k\,, \qquad k = 1, 2, \dots m. \tag{8}$$

Remark 2.2. It has been pointed out in [2] that the condition (6) can be replaced by the less restrictive

$$f(x) < F(x)\,, \qquad 0 < x < X_m. \tag{9}$$

We shall need this condition to prove some monotonicity results and inequalities.

The following important result on the monotonicity of $\dfrac{j_{\nu k}}{\nu}$, due to Makai [39], is based on the Sturm theorem.

Theorem 2.3. *For $k = 1, 2, \dots$ and $\nu > 0$ let $j_{\nu k}$ denote the k-th zero of the Bessel function $J_\nu(x)$ of the first kind. Then*

$$\frac{j_{\nu k}}{\nu} \qquad \textit{decreases as } \nu \textit{ increases},\ \nu > 0,\ k = 1, 2, \dots.$$

Proof. We use the fact that the function

$$y_\nu(x) = x^{1/2}\, J_\nu\left(j_{\nu k}\, x^{1/(2\nu)} \right)$$

satisfies

$$y_\nu'' + p_\nu(x)\, y_\nu = 0$$

where

$$p_\nu(x) = \left\{ \frac{j_{\nu k}}{2\nu}\, x^{\frac{1}{2\nu}-1} \right\}^2 .$$

This follows from the differential equation [47]

$$x^2\, y'' + x\, y' + \left(x^2 - \nu^2\right)\, y = 0$$

by suitable changes of variables.

Now suppose that for some μ and ν, with $0 < \mu < \nu$, we have

$$\frac{j_{\mu k}}{\mu} \leq \frac{j_{\nu k}}{\nu}. \tag{10}$$

The functions $y_\mu(x)$ and $y_\nu(x)$ both vanish at $x = 0$ and both have their k-th positive zero at $x = 1$. Moreover, they satisfy the condition (7) of the Sturm theorem and by (10) $p_\mu(x) < p_\nu(x)$, $0 < x < 1$. Thus an application of the Sturm comparison theorem shows that the k-th zero of $y_\nu(x)$ occurs before the k-th zero of $y_\mu(x)$. This contradiction shows that (7) cannot hold and so $j_{\nu k}/\nu$ decreases as ν increases, $0 < \nu < \infty$. □

Remark 2.4. The reader should observe that Makai used the Sturm theorem not in the usual way, but in an indirect way. In the proof of Theorem 2.2 we know that the k-th zero is exactly at $x = 1$ and we use this fact to prove that the function is monotonic. This is the only indirect application of Sturm theorem that we know.

We recall that this result is essential in the proof of concavity of $j_{\nu k}$, with respect to $\nu > 0$ and of the convexity of $j^2_{\nu k}$, $\nu > 0$. These properties have been conjectured in [42] and proved in [32], [4], [6], [7]. The proofs of Lewis and Muldoon use, among other things, the Hellmann-Feynmann theorem of quantum chemistry.

The second application of Sturm theorem that now we present is concerned with a determinantal inequality whose elements are the zeros of cylinder functions. L. Lorch proved the determinantal inequality

$$\begin{vmatrix} c_{\nu k} & c_{\nu,k+1} \\ c_{\nu,k+1} & c_{\nu,k+2} \end{vmatrix} < 0 \qquad \text{for } \nu \geq 0,\ k = 1, 2, \ldots . \tag{11}$$

It is possible to obtain a more general result, using only the Sturm comparison theorem. This is as follows.

Theorem 2.5. *For $\nu \geq 0$, $k = 1, 2, \ldots$ and $m = 1, 2, \ldots$ let $a_{\nu m}$ and $b_{\nu k}$ be the zeros of the cylinder functions $C_\nu(x)$ and $Z_\nu(x)$, respectively. Suppose that for some $\varepsilon \geq 0$ and for some k and m, $b_{\nu+\varepsilon,k} < a_{\nu m}$. Then*

$$\begin{vmatrix} b_{\nu+\varepsilon,k} & b_{\nu+\varepsilon,k+1} \\ a_{\nu m} & a_{\nu,m+1} \end{vmatrix} < 0. \tag{12}$$

Proof. The function $C_\nu(e^x)$ and $Z_\nu(e^x)$ are the solutions of the differential equation

$$y'' + p(x;\nu)\,y = 0 \tag{13}$$

where

$$p(x;\nu) = e^{2x} - \nu^2.$$

Let

$$h = \log a_{\nu m} - \log b_{\nu+\varepsilon,k} > 0.$$

It is immediate to see that

$$p(x-h,\nu+\varepsilon) < p(x,\nu), \qquad \varepsilon \geq 0,\ h > 0,\ \nu \geq 0$$

and that $Z_{\nu+\varepsilon}(x-h)$ is a solution of

$$z'' + p(x-h,\nu+\varepsilon)\,z = 0. \tag{14}$$

Both the functions $C_\nu(e^x)$ and $Z_{\nu+\varepsilon}(x-h)$ are zero at $x = \log a_{\nu m} = \log b_{\nu+\varepsilon,k} + h$. Thus we can apply the Sturm comparison theorem. Its application to equations (13) and (14) shows that $\log a_{\nu,m+1} < \log b_{\nu+\varepsilon,k+1} + h$ and recalling the value of h, we get

$$\frac{a_{\nu,m+1}}{a_{\nu m}} < \frac{b_{\nu+\varepsilon,k+1}}{b_{\nu+\varepsilon,k}}$$

which is equivalent to (12). □

Remark 2.6. We observe that in the proof of Theorem 2.3 it is not necessary to precise if the functions $C_\nu(e^x)$ and $Z_\nu(x)$ are linearly independent or not.

Remark 2.7. By (12) we can deduce some interesting particular cases. For example with $\varepsilon = 0$, $a_{\nu k} = c_{\nu k}$, $b_{\nu m} = c_{\nu m}$, $m = k+1$, the restriction $b_{\nu+\varepsilon,k} < a_{\nu m}$ becomes $c_{\nu k} < c_{\nu,k+1}$ which is clearly true. In this case by (12) we find

$$\frac{c_{\nu,k+2}}{c_{\nu,k+1}} < \frac{c_{\nu,k+1}}{c_{\nu k}} \qquad k = 1,2,\ldots,\ \nu \geq 0$$

which is the Lorch's inequalities (11).

A more general result has been established by Á. Elbert and A. Laforgia in [8]. Precisely, using the notation $j_{\nu\kappa}$ introduced above to denote the zeros of the cylinder function, they proved the following theorem.

Theorem 2.8. *For ε, δ, h, $r \geq 0$, let T be the determinant defined by*

$$T = \begin{vmatrix} j_{\nu\kappa} & j_{\nu+\delta,\kappa+h} \\ j_{\nu+\varepsilon,\kappa+r} & j_{\nu+\delta+\varepsilon,\kappa+h+r} \end{vmatrix}.$$

If $\varepsilon + r > 0$ and $h + \delta > 0$, then $T < 0$.

We shall not give the proof of this theorem. We say only that this is not based on the Sturm theorem, but on the integral formula (3). By (3) it is possible to derive several monotonicity, concavity (convexity) properties and as a consequence of these, Theorem 2.4. We mention here just one of these results.

Theorem 2.9. *The function* $j_{\nu\kappa}$ *is concave with respect to* κ *if* $\nu \geq \dfrac{1}{2}$ *and convex if* $0 \leq \nu \leq \dfrac{1}{2}$.

We conclude this section with the following remark. The determinants considered in this section are called *Turánians*. In fact P. Turán proved the inequality

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} \leq 0, \qquad -1 \leq x \leq 1$$

where $P_n(x)$ denotes the Legendre polynomial of degree n and where equality holds if and only if $x = \pm 1$. Similar properties have been proved by many authors, for many other special functions. Thus, Karlin and Szegö named these inequalities *Turánians*.

3. A convexity result

J.T. Lewis and M.E. Muldoon [32] proved the convexity of the function $j_{\nu 1}^2$, for $\nu \geq 3$. Á. Elbert and A. Laforgia proved the following more general result.

Theorem 3.1. *Let the function* $j_{\nu\kappa}$ *be defined as above and let*

$$\kappa_0 = \inf\left\{\kappa > 0 : j'_{\nu\kappa} = \frac{d}{d\nu} j_{\nu\kappa} > 1, for\, all\, \nu \geq 0\right\}.$$

Then, $j_{\nu\kappa}^2$ *is a convex function of* ν *for* $\nu \geq 0$ *and for every* $\kappa \geq \kappa_0$.

Proof. It is sufficient to show that

$$\left(\frac{j^2}{2}\right)'' = j'^2 + jj'' > 0. \tag{15}$$

Using the differential equation (3) we have by differentiation

$$\begin{aligned} j'' = {} & 2j' \int_0^\infty K_0\,(2j\sinh t)\, e^{-2\nu t} dt + 2j \int_0^\infty K'_0\,(2j\sinh t)\, 2j' \sinh t e^{-2\nu t} dt \\ & -4j \int_0^\infty K_0\,(2j\sinh t)\, t e^{-2\nu t} dt. \end{aligned} \tag{16}$$

In view of (3) we can write the three terms on the right-hand side of (16) in the following form

$$j'' = \frac{j'^2}{j} + I_1 - I_2. \tag{17}$$

By substitution $u = 2j\sinh t$ the integral I_1 becomes

$$I_1 = 2j' \int_0^\infty K_0'(u)\phi\left(\frac{u}{2j}\right) du,$$

where

$$\phi(x) = \frac{xe^{-2\nu \operatorname{arc\,sinh} x}}{\sqrt{1+x^2}}.$$

An integration by part gives

$$I_1 = 2j'\left[K_0(u)\phi\left(\frac{u}{2j}\right)\right]_0^\infty - \frac{j'}{j}\int_0^\infty K_0(u)\phi'\left(\frac{u}{2j}\right) du, \tag{18}$$

with

$$\phi'(x) = \frac{1 - 2\nu x\sqrt{1+x^2}}{(1+x^2)^{3/2}}\, e^{-2\nu \operatorname{arc\,sinh} x}, \tag{19}$$

and then

$$\phi'(x) < \frac{e^{-2\nu \operatorname{arc\,sinh} x}}{(1+x^2)^{3/2}} < 1, \qquad x > 0,\ \nu \geq 0.$$

Recalling that

$$K_0(x) = \begin{cases} O(\log(1/x)), & x > 0, x \sim 0, \\ o(e^{-x}), & x \gg 1, \end{cases}$$

we have that the first term in the right-hand side of (18) is zero. Then we get

$$I_1 = -\frac{j'}{j}\int_0^\infty K_0(u)\phi'\left(\frac{u}{2j}\right) du. \tag{20}$$

Similarly, for I_2 we have

$$\begin{aligned} I_2 &= 2\int_0^\infty K_0(u)\frac{\operatorname{arc\,sinh}(u/2j)}{\sqrt{1+u^2/4j^2}}\, e^{-2\nu \operatorname{arc\,sinh}(u/2j)} du \\ &= \frac{1}{j}\int_0^\infty K_0(u)u\psi\left(\frac{u}{2j}\right) du, \end{aligned} \tag{21}$$

where

$$\psi(x) = \frac{1}{\sqrt{1+x^2}}\frac{\operatorname{arc\,sinh} x}{x}\, e^{-2\nu \operatorname{arc\,sinh} x}.$$

It is easy to see that

$$\psi(x) < \psi(0) = 1, \qquad x > 0,$$

hence

$$I_2 < \frac{1}{j}\int_0^\infty K_0(u)udu = \frac{1}{j}, \tag{22}$$

where the value of the integral may by found from [47, p. 388]. By (17), taking into account (20), (21), (22) and $j' > 1$ we have in (15)

$$\left(\frac{j^2}{2}\right)'' > j'^2 - j'\int_0^\infty K_0(u)\phi'\left(\frac{u}{2j}\right)du,$$

and therefore it is sufficient to show that

$$I_3 = j' - \int_0^\infty K_0(u)\phi'\left(\frac{u}{2j}\right)du > 0.$$

By (3) and (19) we have

$$I_3 = \int_0^\infty K_0(u)\frac{e^{-2\nu \operatorname{arcsinh}(u/2j)}}{\sqrt{1+u^2/4j^2}}\left[1 + \frac{1-2\nu(u/2j)\sqrt{1+u^2/4j^2}}{1+u^2/4j^2}\right]du,$$

where the quantity between the brackets is clearly positive for $u > 0$. This completes the proof of the convexity of $j^2_{\nu\kappa}$ for $\kappa \geq \kappa_0$ and $\nu \geq 0$. □

Corollary 3.2. *In the particular case $\kappa \equiv k = 1, 2, \ldots$, the function $j^2_{\nu\kappa}$ is convex $\nu \geq 0$.*

We can observe that, since $j^2_{\nu\kappa}$ is convex, the graph of $j^2_{\nu\kappa}$ lies below the chord joining the points $(0, j^2_{0\kappa})$ and $(\nu^*, j^2_{\nu^*\kappa})$. This gives the inequality

$$\frac{j^2_{\nu\kappa} - j^2_{0\kappa}}{\nu} < \frac{j^2_{\nu^*\kappa} - j^2_{0\kappa}}{\nu^*}, \qquad 0 < \nu < \nu^*,$$

i.e., the function $(j^2_{\nu\kappa} - j^2_{0\kappa})/\nu$ increases as ν increases, for $\nu > 0$. Next we consider the chord joining the points $(0, j^2_{0\kappa})$ and $(1/2, j^2_{1/2,\kappa})$ on the graph of $j^2_{\nu\kappa}$ as a function of ν. The convexity of the graph gives

$$j^2_{\nu\kappa} < j^2_{0\kappa} + 2\nu[\kappa^2\pi^2 - j^2_{0\kappa}], \qquad 0 < \nu < 1/2,$$

where the inequality becomes equality only for $\nu = 0$ and $\nu = 1/2$. Similarly, by the convexity of $j^2_{\nu\kappa}$ it follows that

$$j^2_{\nu\kappa} > j^2_{0\kappa} + 2j_{0\kappa}\nu\left[\frac{dj_{\nu\kappa}}{d\nu}\right]_{\nu=0}, \qquad \nu > 0,\ \kappa \geq \kappa_0,$$

and since $j'_{\nu\kappa} > 1$ for $\kappa \geq \kappa_0$,

$$j_{\nu\kappa}^2 > j_{0\kappa}^2 + 2\nu j_{0\kappa}, \qquad \nu > 0.$$

The convexity of $j_{\nu\kappa}^2$ can be used to find many other inequalities too.

Finally one might ask the natural question, whether the validity of the convexity could be extended to the whole domain of the definition of $j_{\nu\kappa}$, i.e., to the interval $(-\kappa,\infty)$. Let us consider only the cases when κ is a natural number, i.e., $\kappa \equiv k = 1,2,\ldots$. Then for the zeros $j_{\nu k}$ of the Bessel function $J_\nu(x)$ we have [47, p. 8]

$$\begin{aligned}
0 &= (\nu+k)\Gamma(\nu+1)\left(\frac{j_{\nu k}}{2}\right)^{-\nu} J_\nu(j_{\nu k}) \\
&= (\nu+k)\left[1-\frac{(j_{\nu k}/2)^2}{1!(\nu+1)}+\cdots+(-1)^{k-1}\frac{(j_{\nu k}/2)^{2(k-1)}}{(k-1)!(\nu+1)\cdots(\nu+k-1)}\right] \\
&= (-1)^k\frac{(j_{\nu k}/2)^{2k}}{k!(\nu+1)\cdots(\nu+k-1)}\left[1-\frac{(j_{\nu k}/2)^2}{(k+1)(\nu+k+1)}+\cdots\right].
\end{aligned}$$

Hence in the right neighborhood of $\nu = -k$ we have

$$\begin{aligned}
\frac{(j_{\nu k}/2)^{2k}}{\nu+k} &= k!(k-1)!(1-\varepsilon)(1-\frac{\varepsilon}{2})\cdots(1-\frac{\varepsilon}{k-1}) \\
&\times\frac{1+\frac{(j_{\nu k}/2)^2}{k-1-\varepsilon}+\cdots+\frac{(j_{\nu k}/2)^{2(k-1)}}{(k-1)!(k-1-\varepsilon)\cdots(1-\varepsilon)}}{1-\frac{(j_{\nu k}/2)^2}{(k+1)(1+\varepsilon)}+\cdots},
\end{aligned} \tag{23}$$

where $\varepsilon = \nu + k$.

Letting $\varepsilon \to 0$ and $j_{\nu k} \to 0$, we obtain

$$\lim_{\nu\to -k+0}\frac{(j_{\nu k}/2)^{2k}}{\nu+k} = k!(k-1)!.$$

We can write this relation in the form

$$\left(\frac{j_{\nu k}}{2}\right)^2 = [k!(k-1)!(\nu+k)]^{1/k}[1+o(1)], \qquad \nu\to -k.$$

In the case $k = 2,3,\cdots$ by (23), we have the more precise approximation

$$\frac{(j_{\nu k}/2)^{2k}}{\nu+k} = k!(k-1)!\left\{1+\frac{2k}{k^2-1}[k!(k-1)!\varepsilon]^{1/k}[1+o(1)]\right\}.$$

Hence

$$\left(\frac{j_{\nu k}}{2}\right)^2 = [k!(k-1)!(\nu+k)]^{1/k}+\frac{2}{k^2-1}[k!(k-1)!(\nu+k)]^{2/k}[1+o(1)],$$
$$\nu\to -k,\ k = 2,3,\ldots.$$

For $k = 1$ we get

$$\left(\frac{j_{\nu 1}}{2}\right)^2 = \nu + 1 + \frac{1}{2}(\nu+1)^2[1+o(1)], \qquad \nu \to -1.$$

4. Higher monotonicity properties

L. Lorch and P. Szegö [37] proved that for $|\nu| > 1/2$, we have

$$(-1)^n \triangle^{n+1} c_{\nu k} > 0, \qquad n = 0, 1, \dots, \; k = 1, 2, \dots. \tag{24}$$

This result has been generalized and extended in [35], [36]. L. Gori, A. Laforgia and M.E. Muldoon [14] proved that (24) remains valid when $c_{\nu k}$ is replaced by $j_{\nu\kappa}$ with a continuous variable κ and the difference operator is replaced by a derivative operator. We mention here some of their results.

Theorem 4.1. *Let $j_{\nu\kappa}$ be defined as above. Then, for $\nu > \mu \geq 1/2$,*

$$(-1)^n D_\kappa^n \left(\log[j_{\nu\kappa}/j_{\mu\kappa}]\right) > 0, \qquad \kappa > 0, \; n = 0, 1, \dots$$

and, in particular, for $\nu > 1/2$,

$$(-1)^n D_\kappa^n (\log[j_{\nu\kappa}/(\kappa\pi)]) > 0, \qquad \kappa > 0, \; n = 0, 1, \dots.$$

Theorem 4.2. *Let $j_{\nu\kappa}$ be defined as above. Then, for $\nu > \mu \geq 1/2$,*

$$(-1)^n D_\kappa^n \left(\log[D_\kappa j_{\nu\kappa}/D_\kappa j_{\mu\kappa}]\right) > 0, \qquad \kappa > 0, \; n = 0, 1, \dots$$

and, in particular, for $\nu > 1/2$,

$$(-1)^n D_\kappa^n (\log[D_\kappa j_{\nu\kappa}/\pi]) > 0, \qquad \kappa > 0, \; n = 0, 1, \dots.$$

Corollary 4.3. *Let $j_{\nu\kappa}$ be defined as above. Then, for $\nu > 1/2$,*

$$(-1)^n D_\kappa^{n+1} (j_{\nu\kappa}) > 0, \qquad \kappa > 0, \; n = 0, 1, \dots.$$

It is not immediate to derive *higher* order monotonicity results, for $0 \leq \nu < 1/2$. In [35] it was conjectured that the *higher* results (24) should be replaced by

$$(-1)^n \triangle^{n+2} c_{\nu k} > 0, \qquad n = 0, 1, \dots, \; k = 1, 2, \dots. \tag{25}$$

In [40] it was proved (25) holds for $1/3 \leq |\nu| < 1/2$. However, for $0 \leq \nu < 1/2$ we can establish some *simple* monotonicity results.

Theorem 4.4. *Let $j_{\nu\kappa}$ be defined as above. Then, if $0 \le \mu < \nu \le 1/2$, the positive function $j_{\nu\kappa}/j_{\mu\kappa}$ decreases to 1 as κ increases on $(0,\infty)$. If $0 \le \mu < 1/2$, then $j_{\mu\kappa}/(\kappa\pi)$ increases with κ, $0 < \kappa < \infty$.*

Theorem 4.5. *Let $j_{\nu\kappa}$ be defined as above. Then, if $0 \le \mu < \nu \le 1/2$, the positive function $D_\kappa j_{\nu\kappa}/D_\kappa j_{\mu\kappa}$ decreases to 1 as κ increases on $(0,\infty)$. If $0 \le \mu < 1/2$, then $D_\kappa j_{\mu\kappa}$ increases with κ, $0 < \kappa < \infty$.*

Many other results of this kind and on the derivative od a zero with respect to κ can be found in [14].

5. Zeros of generalized Airy functions

The generalized Airy functions are nontrivial solutions on $0 \le x < \infty$ of the equation

$$y'' + x^\alpha y = 0 \tag{26}$$

where $\alpha > 0$. When $\alpha = 1$, equation (26) is the Airy equation and its solutions are the Airy functions. In [25] the Authors proved the conjecture formulated by M.S.P. Eastham that the first positive zero $a_{\alpha 1}$ of a solution of (26), with $y(0) = 0$, decreases as α increases. Despite the very simple form of (26) it does seem to be possible an approach to the problem by applying the Sturm comparison theorem. However, (26) is connected to the Bessel equation

$$t^2 \frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 - \nu^2)u = 0,$$

by means of the transformations

$$y(x) = x^{1/2}u(t), \qquad t = 2\nu x^{1/(2\nu)},$$

where $\nu = 1/(\alpha + 2)$. Thus we can refer to the results about the monotonicity, concavity, etc., with respect to order of zeros of Bessel functions. In particular we can prove the following result

Theorem 5.1. *The function $[c_{\nu k}/(2\nu)]^{2\nu}$ increases as ν increases, $0 < \nu \le 1/2$, $k = 2, 3, \ldots$. In case $c_{\nu k} = j_{\nu k}$ the result holds also for $k = 1$.*

Corollary 5.2. *For each $k = 2, 3, \ldots$ $a_{\alpha k}$ decreases to 1 as α increases, $0 < \alpha < \infty$. If $y(0) = 0$, this decrease holds also for $a_{\alpha 1}$.*

An accurate investigation on the properties of generalized Airy functions can be found in [11].

6. Zeros of $K_{i\nu}(x)$

It is known that for real x the function $K_\mu(x)$ has no zeros if μ is real or complex. But, when μ is a pure imaginary number, i.e. $\mu = i\,\nu$, the function $K_\mu(x)$ vanishes at $+\infty$ and has infinitely many positive zeros whose only point of accumulation is $x = 0$, (see [13] and [15] for further information). These zeros occur in certain physical problems such as in the determination of the bound states for an inverse square potential with hard core in Schrödinger equation.

For $k = 1, 2, \ldots$ we denote by $x_k(\nu)$ the positive zeros of the modified Bessel function $K_{i\nu}(x)$ in decreasing order:

$$x_1(\nu) > x_2(\nu) > \cdots > x_n(\nu) > x_{n+1}(\nu) > \cdots > 0$$

and prove the following result, [20].

Theorem 6.1. *For $\nu > 0$ let $x_k(\nu)$ be the k-th positive zero of the modified Bessel function $K_{i\nu}(x)$ of purely imaginary order. Then*

$$x_{k-1}(\nu) - x_k(\nu) < x_{k-2}(\nu) - x_{k-1}(\nu).$$

Proof. We consider the differential equation

$$y'' + P(x;\nu)\, y = 0$$

where

$$P(x;\nu) = -1 + \frac{\frac{1}{4} + \nu^2}{x^2},$$

satisfied by $y_\nu(x) = x^{1/2}\, K_{i\nu}(x)$, [1, p. 377], and the equation

$$z'' + P(x-h;\nu)\, z = 0,$$

where $h = x_{k-1}(\nu) - x_k(\nu)$, satisfied by $y_\nu(x-h)$. It is immediate to check that $P(x;\nu) < P(x-h;\nu)$ and that $y_\nu(x)$ and $y_\nu(x-h)$ both are zero at $x_{k-1}(\nu)$. Thus an application of Sturm theorem gives that the next zero $x_{k-1}(\nu)$ of $y_\nu(x-h)$ occurs before the next zero $x_{k-2}(\nu)$ of $y_\nu(x)$. Recalling the value of h, this shows that

$$x_{k-1}(\nu) - x_k(\nu) < x_{k-2}(\nu) - x_{k-1}(\nu). \qquad \square$$

7. Concluding remarks

We conclude this paper with some remarks about the results on the zeros of special functions. In the paper we referred to the results on the zeros of Bessel functions. Similar properties can be investigated also for the zeros of orthogonal polynomials. For example we mention here the result that $\lambda x_{nk}^{(\lambda)}$ increases for $\lambda > 0$, proved for $0 < \lambda < 1$ and conjectured for any positive λ in [19]. For a proof of the conjecture see [12]. The Sturm comparison theorem has been an useful tool also in the proof of Turán-type inequalities for the zeros of Laguerre polynomials [21]. The same theorem was useful also in the proof of several inequalities for the positive zeros $x_{nk}^{(\lambda)}$ of ultraspherical polynomials. These inequalities are consequence of a general result based on the Sturm theorem, established without serious restrictions on the parameter λ [10]. Finally we mention the papers [30], [26], [27], [31], [28], [29] where the reader can find some recent results about Turánians.

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