

# Zeros of regular functions and polynomials of a quaternionic variable

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## Abstract

The fundamental elements of a new theory of regular functions of a quaternionic variable have been recently developed, following an idea of Cullen. In this paper we present a detailed study of the structure of the zero set of Cullen-regular functions. We prove that the zero sets of the functions under investigation consist of isolated points or isolated 2-spheres, in the 4-dimensional real space of quaternions. Moreover, the zeros of a regular function can be factored by means of a non-standard product. The Fundamental Theorem of Algebra for quaternions, and the approach here adopted lead, in particular, to a deeper insight of the geometric and algebraic properties of the zero sets of polynomials with quaternionic coefficients.

## 1 Introduction

Let  $\mathbb{H}$  denote the skew field of real quaternions. Its elements are of the form  $q = x_0 + ix_1 + jx_2 + kx_3$  where the  $x_l$  are real, and  $i, j, k$ , are imaginary units (i.e. their square equals  $-1$ ) such that  $ij = -ji = k$ ,  $jk = -kj = i$ , and  $ki = -ik = j$ . The richness of the theory of holomorphic functions of one complex variable, along with motivations from physics, aroused a natural interest in a theory of quaternion valued functions of a quaternionic variable. In fact, in the last century, several interesting theories have been introduced. The best known is the one due to Fueter, [3, 4, 5], who defined the differential operator

$$\frac{\partial}{\partial \bar{q}} = \frac{1}{4} \left( \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3} \right),$$

now known as the Cauchy-Fueter operator, and defined the space of regular functions as the space of solutions of the equation associated to this operator. All regular functions are harmonic and Fueter proved that this definition led to close analogues of Cauchy’s theorem, Cauchy’s integral formula and the Laurent expansion. This theory is extremely successful and it is now very well developed, in many different directions. We refer the reader to [14] for the basic features of these functions. Recent work in this subject includes [9, 1], and references therein. For what concerns the zero set of a Fueter-regular function, we remark here that it does not necessarily consist of isolated points, and that its real dimension can be 0, 1, 2 (or 4).

Quite recently, G. Gentili and D. C. Struppa offered an alternative definition and theory of regularity for functions of a quaternionic variable, inspired by an idea of Cullen [2]. Cullen-regular functions are not harmonic in general. This new theory allows the study of natural power series (and polynomials) with quaternionic coefficients, which is excluded when the Fueter approach is followed. A description of this theory can be found in [6, 7]. In the same papers a study of the first properties of the zero set of Cullen-regular functions is performed.

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In order to present the definition of Cullen-regularity, we will start by denoting with  $\mathbb{S}$  the 2-dimensional sphere of imaginary units of  $\mathbb{H}$ , i.e.  $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ . The definition given by Cullen can then be rephrased as follows.

**Definition 1.1.** *Let  $\Omega$  be a domain in  $\mathbb{H}$ . A real differentiable function  $f : \Omega \rightarrow \mathbb{H}$  is said to be  $C$ -regular if, for every  $I \in \mathbb{S}$ , its restriction  $f_I$  to the complex line  $L_I = \mathbb{R} + \mathbb{R}I$  passing through the origin and containing 1 and  $I$  is holomorphic on  $\Omega \cap L_I$ , i.e. if for every  $I$  in  $\mathbb{S}$ ,*

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0,$$

on  $\Omega \cap L_I$ .

Throughout the paper, since no confusion can arise, we will refer to  $C$ -regular functions as regular functions tout court. Since for all  $n \in \mathbb{N}$  and all  $I \in \mathbb{S}$  we have

$$\frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) (x + yI)^n = 0$$

then, by definition, the monomial  $M(q) = q^n$  is regular. Since addition and right multiplication by a constant preserve regularity, all natural polynomials of the form  $P(q) = q^m a_m + \dots + q a_1 + a_0$ , with  $m \in \mathbb{N}$  and  $a_j \in \mathbb{H}$  ( $j = 0, \dots, m$ ) are regular. As observed in [6, 7], for each quaternionic power series

$$f(q) = \sum_{n=0}^{\infty} q^n a_n$$

there exists a ball  $B(0, R) = \{q \in \mathbb{H} : |q| < R\}$  such that  $f$  converges absolutely and uniformly on the compact subsets of  $B(0, R)$  (where its sum defines a regular function) and diverges in  $\mathbb{H} \setminus \overline{B(0, R)}$ .

For regular functions, a notion of derivative can be introduced.

**Definition 1.2.** *Let  $\Omega$  be a domain in  $\mathbb{H}$ , and let  $f : \Omega \rightarrow \mathbb{H}$  be a regular function. The (Cullen) derivative of  $f$ ,  $\frac{\partial f}{\partial q}$ , is defined as follows:*

$$\frac{\partial f}{\partial q}(q) = \begin{cases} \partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI) & \text{if } q = x + yI \text{ with } y \neq 0 \\ \frac{\partial f}{\partial x}(x) & \text{if } q = x \text{ is real} \end{cases}$$

As explained in [6, 7], this definition of derivative is well posed because it only applies to regular functions. It turns out that regular functions defined on domains containing the origin of  $\mathbb{H}$  can be expanded in power series. Namely, if  $B(0, R)$  is the open ball of  $\mathbb{H}$  centered at 0, with radius  $R > 0$ , then we have the following result:

**Theorem 1.3.** *If  $f : B(0, R) \rightarrow \mathbb{H}$  is regular, then it has a series expansion of the form*

$$f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \frac{\partial^n f}{\partial q^n}(0)$$

converging on  $B(0, R)$ . In particular  $f$  is  $C^\infty$  on  $B(0, R)$ .

Roughly speaking, there is a correspondence between quaternionic power series centered at 0 and regular functions on domains containing the origin of  $\mathbb{H}$ . In [6, 7] the first fundamental results of the theory of (Cullen) regular functions are proved: the identity principle, the maximum modulus principle, the Cauchy representation formula, the Liouville theorem and the Morera theorem. A version of the Schwarz lemma opens possible advances in the study of the geometry of the unit ball of  $\mathbb{H}$ , of the four dimensional analog of the Siegel right-half plane (biregular to the unit ball via the analogous of a Cayley), and their transformations. Finally, we recall the statement of a first (purely algebraic) property of the zeros of regular functions which is proven in [6, 7]:

**Theorem 1.4.** *Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a quaternionic power series converging in  $B(0, R)$  and let  $x, y \in \mathbb{R}$  be such that  $y \neq 0, x^2 + y^2 < R^2$ . If there exist two distinct imaginary units  $I, J \in \mathbb{S}$  such that  $f(x + yI) = 0 = f(x + yJ)$  then  $f$  vanishes on the whole 2-sphere  $x + y\mathbb{S} = \{x + yL : L \in \mathbb{S}\}$ .*

The same result was previously proven for polynomials in [12]. Theorem 1.4 enlightens a symmetry property of the zeros of regular functions, but it does not predict the topological features of the zero set of such functions.

We begin this paper by proving the following topological property of the zero set of regular functions, which urges a comparison with the case of holomorphic functions of one complex variable. Namely we prove that

**Theorem 2.4** (Structure of the Zero Set). *Let  $f$  be a regular function on an open ball  $B(0, R)$  centered in the origin of  $\mathbb{H}$ . If  $f$  is not identically zero then its zero set consists of isolated points or isolated 2-spheres of the form  $S = x + y\mathbb{S}$ , for  $x, y \in \mathbb{R}$ ,  $y \neq 0$ .*

The above result is proven for the polynomial case in [12], by means of simpler techniques. It naturally leads to the formulation of an identity principle which generalizes the one stated in [6, 7]:

**Theorem 2.5** (Strong Identity Principle). *Let  $f, g : B(0, R) \rightarrow \mathbb{H}$  be regular functions. If there exist  $x, y \in \mathbb{R}$  such that  $S = x + y\mathbb{S} \subseteq B(0, R)$  and a subset  $\mathcal{T} \subseteq B(0, R) \setminus S$  having an accumulation point in  $S$  such that  $f \equiv g$  on  $\mathcal{T}$ , then  $f \equiv g$  on the whole domain of definition  $B(0, R)$ .*

Remark that  $S = x + y\mathbb{S}$  is a 2-sphere if  $y \neq 0$ , a real singleton  $\{x\}$  if  $y = 0$ . The proof of theorem 2.4 is much harder than the proof of the homologous result in complex analysis, and has a different structure. In fact the factorization property of the zeros of holomorphic functions does not extend to regular functions, due to the lack of commutativity. Nevertheless, the techniques employed to prove theorem 2.4 suggest the use of the following multiplication between regular power series:

**Definition 3.1.** *Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  and  $g(q) = \sum_{n=0}^{+\infty} q^n b_n$  be given quaternionic power series with radii of convergence greater than  $R$ . We define the regular product of  $f$  and  $g$  as the series  $f * g(q) = \sum_{n=0}^{+\infty} q^n c_n$ , where  $c_n = \sum_{k=0}^n a_k b_{n-k}$  for all  $n$ .*

We point out that the sequence of the coefficients of the regular product  $f * g$  is the discrete convolution of the sequences of the coefficients of  $f$  and  $g$ . In the polynomial case, the regular multiplication coincides with the classical multiplication of the polynomial ring over the quaternions,  $\mathbb{H}[X]$ . In terms of the above defined product we obtain a factorization result for the zeros of regular functions (theorem 3.2) and completely describe the zero set of a regular product in terms of the zero sets of the two factors:

**Theorem 3.3** (Zeros of a regular product). *Let  $f, g$  be given quaternionic power series with radii greater than  $R$  and let  $p \in B(0, R)$ . Then  $f * g(p) = 0$  if and only if  $f(p) = 0$  or  $f(p) \neq 0$  and  $g(f(p)^{-1} p f(p)) = 0$ .*

This extends to quaternionic power series the theory presented in [10] for polynomials. In particular, given the power series expansion of any regular function  $f$ , we construct the symmetrization  $f^s$  of  $f$  which has real coefficients and vanishes exactly on the 2-spheres (or singletons)  $x + y\mathbb{S}$  where  $f$  has a zero.

When applied to polynomials, the above results and the fundamental theorem of algebra for quaternions (see section 5 for reference) lead to a factorization theorem which states that

**Theorem 5.2** (Factorization). *Let  $a_0, \dots, a_n \in \mathbb{H}$ ,  $a_n \neq 0$  and  $f(q) = a_0 + qa_1 + \dots + q^n a_n$ . Then there exist points  $p_1, \dots, p_n \in \mathbb{H}$  such that  $f(q) = (q - p_1) * \dots * (q - p_n) c$ , where  $c = a_n$ .*

and to the complete identification of the zeros of polynomials in terms of their factorization. These last results have already been proven in [13] from an algebraic point of view. Our new approach enriches them with a technique to localize the zeros of polynomials.

Finally, the most natural definition of multiplicity leads to the result that the degree of a polynomial might exceed the sum of the multiplicities of its zeros (Proposition 5.6).

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## 2 Structure of the zero set of regular functions

One of the basic properties of holomorphic functions of a complex variable is the discreteness of their zero sets (except for the case where the function vanishes identically). Given a regular quaternionic function  $f$  on a ball  $B(0, R)$ , all of its restrictions  $f_I$  to complex lines  $L_I$  are holomorphic and hence either have a discrete zero set or vanish identically. By the identity principle proven in [6, 7], if  $f_I \equiv 0$  for some  $I \in \mathbb{S}$  then  $f \equiv 0$ . Therefore, the zeros of a non-trivial  $f$  can not accumulate on a single complex line  $L_I$ . However, this does not prevent the zeros of  $f$  from accumulating tout court: as we have seen in theorem 1.4, a regular function may well have a whole 2-sphere  $x + y\mathbb{S}$  of zeros. The result we announced in the introduction as theorem 2.4 tells us that this is the only way the zeros of a regular function can accumulate: every zero of  $f$  is either isolated or part of an isolated 2-sphere of zeros.

In order to prove the desired result, we need to take some preliminary steps. First of all, we describe a necessary and sufficient condition for a quaternionic regular function  $f$  to have a zero at point  $p$ , in terms of the coefficients of the power series expansion of  $f$ . This result is a non-commutative generalization of a well-known property of holomorphic functions of a complex variable: a holomorphic function  $f$  has a zero at point  $p$  if and only if there exists a holomorphic function  $g$  such that  $f(z) = (z - p)g(z)$  for all  $z$  in a neighbourhood of  $p$ .

**Theorem 2.1.** *Let  $\sum_{n=0}^{+\infty} q^n a_n$  be a given quaternionic power series with radius of convergence  $R$  and let  $p \in B(0, R)$ . Then  $\sum_{n=0}^{+\infty} p^n a_n = 0$  if and only if there exists a quaternionic power series  $\sum_{n=0}^{+\infty} q^n c_n$  with radius of convergence  $R$  such that  $a_0 = -pc_0$  and  $a_n = c_{n-1} - pc_n$  for all  $n > 0$ .*

*Proof.* Let  $I \in \mathbb{S}$  be an imaginary unit such that  $p \in L_I$  and let  $J \in \mathbb{S}$  be such that  $I \perp J$ . There exist sequences  $\{\alpha_n\}_{n=0}^{+\infty}, \{\beta_n\}_{n=0}^{+\infty}$  in  $L_I$  such that  $a_n = \alpha_n + \beta_n J$  for all  $n$ . The equation

$$0 = \sum_{n=0}^{+\infty} p^n a_n = \sum_{n=0}^{+\infty} p^n \alpha_n + \sum_{n=0}^{+\infty} p^n \beta_n J$$

is equivalent to  $0 = \sum_{n=0}^{+\infty} p^n \alpha_n = \sum_{n=0}^{+\infty} p^n \beta_n$ . By identifying  $L_I$  with the complex plane  $\mathbb{C}$ , we can consider the two complex power series  $\sum_{n=0}^{+\infty} z^n \alpha_n, \sum_{n=0}^{+\infty} z^n \beta_n$ , whose radii of convergence  $R_1, R_2$  are such that  $\min(R_1, R_2) = R$ . These two series have a zero at  $p$  if and only if there exist complex power series  $\sum_{n=0}^{+\infty} z^n \gamma_n, \sum_{n=0}^{+\infty} z^n \delta_n$  with radii  $R_1, R_2$  such that

$$\begin{aligned} \sum_{n=0}^{+\infty} z^n \alpha_n &= (z - p) \sum_{n=0}^{+\infty} z^n \gamma_n = -p\gamma_0 + \sum_{n=1}^{+\infty} z^n (\gamma_{n-1} - p\gamma_n) \\ \sum_{n=0}^{+\infty} z^n \beta_n &= (z - p) \sum_{n=0}^{+\infty} z^n \delta_n = -p\delta_0 + \sum_{n=1}^{+\infty} z^n (\delta_{n-1} - p\delta_n) \end{aligned}$$

i.e. such that  $\alpha_0 = -p\gamma_0, \beta_0 = -p\delta_0$  and  $\alpha_n = \gamma_{n-1} - p\gamma_n, \beta_n = \delta_{n-1} - p\delta_n$  for all  $n > 0$ . Recalling that  $a_n = \alpha_n + \beta_n J$  and setting  $c_n = \gamma_n + \delta_n J$  for all  $n$ , the latter is equivalent to

$$a_0 = -pc_0, \quad a_n = c_{n-1} - pc_n \quad \text{for all } n > 0.$$

It is now sufficient to remark that the radius of convergence of  $\sum_{n=0}^{+\infty} q^n c_n$  equals  $\min(R_1, R_2) = R$ .  $\square$

The previous result allows our second step towards the proof of theorem 2.4. For any quaternionic power series  $f$ , we are able to construct a new quaternionic power series  $f^s$  in such a way that whenever  $f(x + yI) = 0$  for some  $I \in \mathbb{S}$  we can conclude  $f^s(x + yL) = 0$  for all  $L \in \mathbb{S}$ .

**Definition 2.2.** *Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a given quaternionic power series with radius of convergence  $R$ . We define the symmetrization of  $f$  as the series  $f^s(q) = \sum_{n=0}^{+\infty} q^n r_n$ , where  $r_n = \sum_{k=0}^n a_k \bar{a}_{n-k}$  for all  $n$ .*

It is easy to prove that  $f^s$  also has radius of convergence  $R$ . We also notice that coefficients  $r_n = \sum_{k=0}^n a_k \bar{a}_{n-k}$  all belong to  $\mathbb{R}$ . We now prove the above mentioned relation between the zeros of a series and the zeros of its symmetrization.

**Proposition 2.3.** *Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a given quaternionic power series with radius of convergence  $R$ . If  $x, y \in \mathbb{R}$ ,  $I \in \mathbb{S}$  and  $f(x + yI) = 0$  then  $f^s(x + yL) = 0$  for all  $L \in \mathbb{S}$ .*

*Proof.* Let  $p = x + yI$  be a zero of  $f$ . By theorem 2.1, this implies the existence of a series  $g(q) = \sum_{n=0}^{+\infty} q^n c_n$  with radius  $R$  such that  $a_0 = -pc_0$  and

$$a_n = c_{n-1} - pc_n$$

for all  $n > 0$ . If we set  $r_n = \sum_{k=0}^n a_k \bar{a}_{n-k}$  and  $s_n = \sum_{k=0}^n c_k \bar{c}_{n-k}$  for all  $n$ , the above equality implies by direct computation  $r_0 = |p|^2 s_0, r_1 = -2xs_0 + |p|^2 s_1$  and

$$r_n = s_{n-2} - 2xs_{n-1} + |p|^2 s_n$$

for all  $n > 1$ . From this we get

$$\begin{aligned} f^s(q) &= \sum_{n=0}^{+\infty} q^n r_n = \sum_{n=0}^{+\infty} q^{n+2} s_n - 2x \sum_{n=0}^{+\infty} q^{n+1} s_n + |p|^2 \sum_{n=0}^{+\infty} q^n s_n = \\ &= (q^2 - 2xq + |p|^2) g^s(q) = [(q - x)^2 + y^2] g^s(q), \end{aligned}$$

which gives immediately  $f^s(x + yL) = 0$  for all  $L \in \mathbb{S}$ .  $\square$

We are now ready to prove theorem 2.4. Symmetrization allows us indeed to transform any zero into a ‘‘spherical’’ zero and these zeros can not accumulate: if they did, zeros would accumulate in each complex line  $L_I$  and this is impossible, as we discussed at the beginning of this section. We state our result before giving the detailed proof.

**Theorem 2.4** (Structure of the Zero Set). *Let  $f : B(0, R) \rightarrow \mathbb{H}$  be a regular function and suppose  $f$  does not vanish identically. Then the zero set of  $f$  consists of isolated points or isolated 2-spheres of the form  $S = x + y\mathbb{S}$ , for  $x, y \in \mathbb{R}$ .*

*Proof.* Let  $f : B(0, R) \rightarrow \mathbb{H}$  be any regular function and let  $\mathcal{Z}_f$  be its zero set. Consider any 2-sphere (or singleton)  $S = x + y\mathbb{S} \subseteq B(0, R)$  containing zeros of  $f$ . We already know, by theorem 1.4, that either  $f$  has exactly one zero in  $S$  or  $f$  vanishes at all points of  $S$ . We only have to prove that if  $\mathcal{Z}_f \setminus S$  has an accumulation point in  $S$  then  $f \equiv 0$ .

Let  $p = x + yI \in S$  be such a point: there exists a sequence of zeros of  $f$  not belonging to  $S$ ,  $\{p_n\}_{n=0}^{+\infty} \subseteq \mathcal{Z}_f \setminus S$ , which converges to  $p$ . Consider the power series expansion  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  and its symmetrization  $f^s(q) = \sum_{n=0}^{+\infty} q^n r_n$ : for any given  $n$ , the fact that  $f$  vanishes at  $p_n = x_n + y_n I_n$  implies  $f^s(x_n + y_n J) = 0$  for all  $J \in \mathbb{S}$ . Now identify  $L_I$  with the complex plane  $\mathbb{C}$ : the complex series (with real coefficients)  $\sum_{n=0}^{+\infty} z^n r_n$  is zero at all points  $x_n + y_n I$ , which accumulate at  $p$ . By a well-known property of complex power series, coefficients  $r_n$  must all vanish. It can be easily proven by induction that the vanishing of  $r_n = \sum_{k=0}^n a_k \bar{a}_{n-k}$  for all  $n$  implies the vanishing of all coefficients  $a_n$ . This means  $f \equiv 0$ , as wanted.  $\square$

As a consequence of the previous result, we can strengthen the identity principle proven in [6, 7].

**Theorem 2.5** (Strong Identity Principle). *Let  $f, g : B(0, R) \rightarrow \mathbb{H}$  be regular functions. If there exist  $x, y \in \mathbb{R}$  such that  $S = x + y\mathbb{S} \subseteq B(0, R)$  and a subset  $\mathcal{T} \subseteq B(0, R) \setminus S$  having an accumulation point in  $S$  such that  $f \equiv g$  on  $\mathcal{T}$ , then  $f \equiv g$  on the whole domain of definition  $B(0, R)$ .*

*Proof.* Consider the regular function  $h = f - g : B(0, R) \rightarrow \mathbb{H}$  and its zero set  $\mathcal{Z}_h$ . We know that  $\mathcal{T} \subseteq \mathcal{Z}_h$ , so  $\mathcal{Z}_h \setminus S$  has an accumulation point in  $S$ . By the structure theorem,  $h \equiv 0$ . This implies  $f \equiv g$ , as wanted.  $\square$

### 3 Regular multiplication

The proof of the structure theorem we gave in the previous section required quite a lot of work, if compared to the proof of the analogous result in complex analysis. The fact of the matter is that the factorization property of the zeros of holomorphic complex functions is substituted by theorem 2.1, which is apparently a weaker result because of the non-commutativity of multiplication in  $\mathbb{H}$ . This makes handling the zeros harder than in the complex case. In this section we show that, using a different notion of multiplication between regular functions, theorem 2.1 can be turned into a factorization result.

**Definition 3.1.** Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  and  $g(q) = \sum_{n=0}^{+\infty} q^n b_n$  be given quaternionic power series with radii of convergence greater than  $R$ . We define the regular product of  $f$  and  $g$  as the series  $f * g(q) = \sum_{n=0}^{+\infty} q^n c_n$ , whose coefficients  $c_n = \sum_{k=0}^n a_k b_{n-k}$  are obtained by discrete convolution from the coefficients of  $f$  and  $g$ .

The regular product of  $f$  and  $g$ , which we denote indifferently as  $f * g$ ,  $f * g(q)$  or  $f(q) * g(q)$ , has radius of convergence greater than  $R$ . It can be easily proven that the regular multiplication  $*$  is an associative, non-commutative operation. We can now restate theorem 2.1 as follows.

**Theorem 3.2.** Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a given quaternionic power series with radius of convergence  $R$  and let  $p \in B(0, R)$ . Then  $f(p) = 0$  if and only if there exists a quaternionic power series  $g(q)$  with radius of convergence  $R$  such that

$$f(q) = (q - p) * g(q). \quad (1)$$

This result would of course be uninteresting if the other zeros of  $f$  did not depend on the zeros of  $g$ . Fortunately, this is not the case: the zeros of a regular product  $f * g$  are strongly related with those of  $f$  and  $g$ .

**Theorem 3.3** (Zeros of a regular product). Let  $f, g$  be given quaternionic power series with radii greater than  $R$  and let  $p \in B(0, R)$ . Then  $f * g(p) = 0$  if and only if  $f(p) = 0$  or  $f(p) \neq 0$  and  $g(f(p)^{-1} p f(p)) = 0$ .

*Proof.* It can be easily proven that if  $g(q) = \sum_{n=0}^{+\infty} q^n b_n$  then  $f * g(q) = \sum_{n=0}^{+\infty} q^n f(q) b_n$ . Hence  $f(p) = 0$  implies  $f * g(p) = 0$  and  $f(p) \neq 0$  implies

$$f * g(p) = f(p) \sum_{n=0}^{+\infty} f(p)^{-1} p^n f(p) b_n = f(p) g(f(p)^{-1} p f(p)),$$

so that  $f * g(p) = 0$  iff  $g(f(p)^{-1} p f(p)) = 0$ . □

In particular, if  $f * g$  has a zero in  $S = x + y\mathbb{S}$  then either  $f$  or  $g$  have a zero in  $S$ . However, the zeros of  $g$  in  $S$  need not be in one-to-one correspondence with the zeros of  $f * g$  in  $S$  which are not zeros of  $f$ .

**Example 3.4.** Let  $I \in \mathbb{S}$  be an imaginary unit. The regular product

$$(q - I) * (q + I) = q^2 + 1$$

has  $\mathbb{S}$  as its zero set, while  $q - I, q + I$  only vanish at  $I, -I$  respectively.

**Example 3.5.** Let  $I, J \in \mathbb{S}$  be different imaginary units and suppose  $I \neq -J$ . The regular product

$$(q - I) * (q - J) = q^2 - q(I + J) + IJ$$

vanishes at  $I$ , but has no other zero in  $\mathbb{S}$ : given any  $L \in \mathbb{S}$ , we get  $L^2 - L(I + J) + IJ = 0$  iff  $L(I + J) = -1 + IJ$  iff  $L(I + J) = I(I + J)$  iff  $L = I$ , since  $I + J \neq 0$ .

## 4 Symmetrization and computation of the zeros

In this section we complete the characterization of the zero set of  $f^s$  in terms of the zero set of  $f$ . This leads to a method to compute the zeros of a quaternionic regular function. The new result on the zeros of  $f^s$  is based on the fact that  $f^s = f * f^c$ , where  $f^c$  is a new series called the regular conjugate of  $f$ .

**Definition 4.1.** Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a given quaternionic power series with radius of convergence  $R$ . We define the regular conjugate of  $f$  as the series  $f^c(q) = \sum_{n=0}^{+\infty} q^n \bar{a}_n$ .

We remark that  $f^c$  also has radius  $R$  and that  $f^s = f * f^c$ . Moreover, we prove the following.

**Proposition 4.2.** Let  $f$  be a given quaternionic power series with radius of convergence  $R$  and let  $x, y \in \mathbb{R}$  be such that  $S = x + y\mathbb{S} \subseteq B(0, R)$ . The zeros of  $f$  in  $S$  are in one-to-one correspondence with those of  $f^c$ .

*Proof.* Since  $(f^c)^c = f$ , we only have to prove that the vanishing of  $f$  at one or all points of  $S$  implies the vanishing of  $f^c$  at one or all points of  $S$  respectively.

Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  and for all  $n \in \mathbb{N}$  let  $s_n, t_n \in \mathbb{R}$  be such that  $(x + yL)^n = s_n + Lt_n$  for all  $L \in \mathbb{S}$ . Then

$$\begin{aligned} f(x + yL) &= \sum_{n=0}^{+\infty} (x + yL)^n a_n = \sum_{n=0}^{+\infty} (s_n + Lt_n) a_n = b + Lc \\ f^c(x + yL) &= \sum_{n=0}^{+\infty} (x + yL)^n \bar{a}_n = \sum_{n=0}^{+\infty} (s_n + Lt_n) \bar{a}_n = \bar{b} + L\bar{c} \end{aligned}$$

for all  $L \in \mathbb{S}$ , letting  $b = \sum_{n=0}^{+\infty} s_n a_n$  and  $c = \sum_{n=0}^{+\infty} t_n a_n$ . If  $f \equiv 0$  on  $S$  then for all  $L \in \mathbb{S}$  we get  $0 = f(x + yL) = f(x - yL)$ . Hence  $0 = b + Lc = b - Lc$  and  $b = c = 0$ , so that  $\bar{b} = \bar{c} = 0$  and  $f^c(x + yL) = 0$  for all  $L \in \mathbb{S}$ . Now suppose  $f$  has exactly one zero in  $S$ , namely  $p = x + yI$ . Then  $c \neq 0$ : if  $c$  vanished then  $0 = f(p) = b + Ic$  would imply  $b = c = 0$  and  $f \equiv 0$  in  $S$ . Hence  $\bar{c} \neq 0$  and from  $0 = f(p) = b + Ic$  we can conclude

$$\begin{aligned} 0 &= \overline{b + Ic} = \bar{b} - \bar{c}I = \bar{b} - (\bar{c}I\bar{c}^{-1})\bar{c} = \\ &= \bar{b} + J\bar{c} = f^c(x + yJ) \end{aligned}$$

where  $J = -\bar{c}I\bar{c}^{-1} \in \mathbb{S}$ . □

We are now ready to study the zero set of  $f^s$ .

**Theorem 4.3.** Let  $f$  be any given quaternionic power series with radius  $R$ . Then  $f^s$  vanishes exactly on the 2-spheres (or singletons)  $x + y\mathbb{S}$  where  $f$  has a zero.

*Proof.* Proposition 2.3 tells us that the zero set of  $f^s$  includes all the 2-spheres (or singletons)  $x + y\mathbb{S}$  on which  $f$  has a zero. Conversely, any zero of  $f^s$  lies on a 2-sphere (or singleton)  $x + y\mathbb{S}$  on which  $f$  has a zero: if  $f^s = f * f^c$  vanishes at  $x + yI$ , then either  $f$  or  $f^c$  have a zero in  $x + y\mathbb{S}$ ; by the previous proposition, this implies that  $f$  has a zero in  $x + y\mathbb{S}$ . □

The above result radically simplifies the computation of the zeros of a given power series  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ . Consider indeed symmetrization  $f^s(q) = \sum_{n=0}^{+\infty} q^n r_n$  and its restriction to a complex line  $L_I$ . This restriction can be identified, as discussed in previous sections, with the complex series (with real coefficients)  $H(z) = \sum_{n=0}^{+\infty} z^n r_n$ . Computing the zeros of the complex function  $H$  immediately determines the zero set of  $f^s$ , hence the real points and 2-spheres where  $f$  has zeros. For any such 2-sphere  $S = x + y\mathbb{S}$  we can compute, as we did proving theorem 4.2, constants  $b, c \in \mathbb{H}$  such that  $f(x + yL) = b + Lc$  for all  $L \in \mathbb{S}$ . If  $b = c = 0$  then  $f$  vanishes at all points of  $S$ , otherwise  $c \neq 0$  and  $f$  has exactly one zero in  $S$ , the point  $p = x + yJ$  with  $J = -bc^{-1} \in \mathbb{S}$ .

**Example 4.4.** Consider the polynomial

$$f(q) = (q - I) * (q - J) = q^2 - q(I + J) + IJ$$

for  $I, J \in \mathbb{S}$  with  $I \neq -J$ . By an easy computation,  $f^s(q) = (q^2 + 1)^2$  which has  $\mathbb{S}$  as its zero set. Hence the zeros of  $f$  are contained in  $\mathbb{S}$ . We already proved that the only zero of  $f$  in  $\mathbb{S}$  is point  $I$ , so  $f$  only vanishes at  $I$ .

It seems pretty natural for  $(q - I) * (q - I)$  to only vanish at  $I$ . On the other hand, the fact that the zero set of  $(q - I) * (q - J)$  be a singleton even when  $I \neq J$  seems very peculiar and suggests a deeper study of quaternionic polynomials.

Before proceeding towards this study, which is the aim of next section, we remark two useful multiplicative properties of regular conjugation and symmetrization. These properties are naturally connected to the relation between the zeros of  $f$  and those of  $f^c, f^s$  and they will prove very useful in the polynomial case.

**Theorem 4.5.** Let  $f, g$  be given quaternionic power series. Then  $(f * g)^c = g^c * f^c$  and

$$(f * g)^s = f^s g^s = g^s f^s. \quad (2)$$

The first property can be proven by direct computation of the coefficients of the series and the second follows.

## 5 Zeros of quaternionic polynomials and multiplicity

This section is dedicated to the study of quaternionic polynomials and their zeros. First of all, we prove that all quaternionic polynomials have a “regular factorization”. Thanks to the results proven in the previous section, we can easily predict the zeros of a polynomial knowing its factorization and vice versa. By defining the concept of multiplicity in the most natural way, we are led to the result that the sum of the multiplicities of the zeros of a polynomial need not equal its degree.

Our factorization result makes use of the fundamental theorem of algebra for quaternions. This theorem is well known and it can be proven by different techniques. We will rephrase here the interesting proof given in [12].

**Theorem 5.1** (Fundamental Theorem of Algebra for Quaternions). A quaternionic polynomial  $a_0 + qa_1 + \dots + q^n a_n$  of degree  $n \geq 1$  has at least one zero in  $\mathbb{H}$ .

*Proof.* Let  $f(q) = a_0 + qa_1 + \dots + q^n a_n$ . Its symmetrization  $f^s(q) = r_0 + qr_1 + \dots + q^{2n} r_{2n}$  is a polynomial of degree  $2n \geq 2$  with real coefficients  $r_m = \sum_{k=0}^m a_k \bar{a}_{m-k} \in \mathbb{R}$ . By the fundamental theorem of algebra for complex polynomials,  $f^s$  must have a zero in  $\mathbb{H}$ . By theorem 4.3,  $f^s$  has zeros if and only if  $f$  has at least one. Thus  $f$  has a zero in  $\mathbb{H}$ , too.  $\square$

An algebraic proof of the same theorem can be found, for instance, in [11]. A recent topological proof, which applies to all division algebras, is given in [8]. We are now ready to prove our factorization result.

**Theorem 5.2.** Let  $a_0, \dots, a_n \in \mathbb{H}$ ,  $a_n \neq 0$  and  $f(q) = a_0 + qa_1 + \dots + q^n a_n$ . Then there exist points  $p_1, \dots, p_n \in \mathbb{H}$  such that

$$f(q) = (q - p_1) * \dots * (q - p_n) c \quad (3)$$

where  $c = a_n$ .

*Proof.* If  $n = 0$  our thesis is obvious. Supposing the theorem holds for all polynomials of degree  $n$ , we will prove it for a polynomial  $f$  of degree  $n + 1$ . By the fundamental theorem of algebra,  $f$  has a zero  $p \in \mathbb{H}$ . By theorem 3.2, there exists a polynomial  $g$  of degree  $n$  such that  $f(q) = (q - p) * g(q)$ . Hence  $g(q) = (q - p_1) * \dots * (q - p_n) c$  for some  $p_1, \dots, p_n, c \in \mathbb{H}$  and the thesis follows.  $\square$



We will now study how many different factorizations a polynomial can have. If  $f(q) = (q - p_1) * \dots * (q - p_n)c$ , supposing  $p_k = x_k + y_k I_k$  for all  $k$ , by theorem 4.5 we get

$$f^s(q) = |c|^2 \prod_{k=1}^n [(q - x_k)^2 + y_k^2].$$

By this formula we can easily remark the following.

**Proposition 5.3.** *Consider two polynomials  $f(q) = (q - p_1) * \dots * (q - p_n)c$ ,  $g(q) = (q - p'_1) * \dots * (q - p'_m)c'$  and suppose  $p_k = x_k + y_k I_k, p'_h = x'_h + y'_h I'_h$  for all  $k, h$ . Then  $f^s = g^s$  if and only if  $n = m$ ,  $|c| = |c'|$  and  $(x_1, y_1), \dots, (x_n, y_n)$  is a permutation of  $(x'_1, y'_1), \dots, (x'_n, y'_n)$ .*

This is, in particular, a necessary condition for  $f$  to equal  $g$ . In order to find a condition which is also sufficient, we focus on the case where  $n = 2$  and  $c = 1$ . Consider indeed a polynomial

$$f(q) = (q - a) * (q - b) = q^2 - q(a + b) + ab.$$

If  $a = \bar{b}$  and  $a, b \in S = x + y\mathbb{S}$  then  $f(q) = (q - x)^2 + y^2$ . Thanks to proposition 5.3, it is easy to prove that  $f(q) = (q - a') * (q - b')$  if and only if  $a', b' \in S$  and  $a' = \bar{b}'$ .

If  $a, b$  lie on the same  $S$ , but  $a \neq \bar{b}$ , then  $f$  can only be factored as  $f(q) = (q - a) * (q - b)$ . Supposing indeed  $f(q) = (q - a') * (q - b')$ , we get  $a', b' \in S$  by proposition 5.3 and we can easily conclude that  $a' = a, b' = b$ .

Now suppose  $a, b$  lie on different 2-spheres (or real singletons)  $S_a, S_b$ . Supposing  $a' \in S_a, b' \in S_b$ , it is easy to prove that  $f(q) = (q - a') * (q - b')$  if and only if  $a' = a, b' = b$  and  $f(q) = (q - b') * (q - a')$  if and only if  $a' = cac^{-1}, b' = bcb^{-1}$  where  $c = a - \bar{b} \neq 0$ . By proposition 5.3, there is no other alternative. So  $f$  has exactly two factorizations:  $f(q) = (q - a) * (q - b)$  and  $f(q) = (q - bcb^{-1}) * (q - cac^{-1})$ .

Recalling that, by theorem 3.2, every zero can be factored ‘‘on the left’’, the three configurations above correspond to different structures of the zero set.

**Theorem 5.4.** *Let  $a, b \in \mathbb{H}$  and  $f(q) = (q - a) * (q - b)$ . If  $a, b$  lie on different 2-spheres (or real singletons) then  $f$  has two zeros,  $a$  and  $(a - \bar{b})b(a - \bar{b})^{-1}$ . If  $a, b$  lie on the same 2-sphere  $S$  but  $a \neq \bar{b}$  then  $f$  only vanishes at  $a$ . Finally, if  $a = \bar{b} \in S$  then the zero set of  $f$  is  $S$ .*

It seems perfectly natural, thanks to the study accomplished in section 2, that some polynomials have as many zeros as their degrees predict and some have a whole 2-sphere instead of a couple of zeros. It also seems natural for the ‘‘regular square’’  $(q - a) * (q - a)$  to just vanish at  $a$ . The very peculiar case is that of a polynomial  $(q - a) * (q - b)$  where  $a, b$  are different, non-conjugate points of the same 2-sphere: the uniqueness of the zero  $a$  does not seem to be justified by multiplicity arguments. We now translate this impression in a more rigorous result. First of all, we define the regular power of a series  $f$  in the most obvious way:

$$f^{*n} = f * \dots * f = *_{i=1}^n f$$

Now we define the multiplicity of a zero.

**Definition 5.5.** *Let  $f(q) = \sum_{n=0}^{+\infty} q^n a_n$  be a given quaternionic power series with radius  $R$ , and let  $p \in B(0, R)$ . We define the multiplicity of  $p$  as a zero of  $f$  and denote by  $m_p(f)$  the largest  $n \in \mathbb{N}$  such that there exists a series  $g$  with  $f(q) = (q - p)^{*n} * g(q)$ .*

Letting  $I \in \mathbb{S}$  be such that  $p \in L_I$ , the equality  $f(q) = (q - p)^{*n} * g(q)$  implies by restriction to the complex line  $L_I$  that  $f_I(z) = (z - p)^n g_I(z)$ . Hence the multiplicity of a zero of  $f$  is well-defined: by a well-known fact in complex analysis, there is just a finite set of natural numbers  $n$  such that  $(z - p)^n$  can be factored from the holomorphic function  $f_I(z)$ .

Conversely, it can be proven that if there exists a complex series (with quaternionic coefficients)  $H(z) = \sum_{n=0}^{+\infty} z^n a_n$  such that  $f_I(z) = (z - p)^n H(z)$ , then  $f(q) = (q - p)^{*n} * g(q)$  with  $g(q) = \sum_{n=0}^{+\infty} q^n a_n$ . Hence the quaternionic multiplicity defined above extends coherently the definition of complex multiplicity. Nevertheless, it leads to the result that

**Proposition 5.6.** *The degree of a polynomial can exceed the sum of the multiplicities of its zeros.*

We conclude with an explicit example, to prove and make clear our last statement. Consider (again) the polynomial

$$f(q) = (q - I) * (q - J) = q^2 - q(I + J) + IJ$$

and suppose  $I, J \in \mathbb{S}$  with  $I \neq J$ ,  $I \neq -J$ . We already proved that the zero set of  $f$  is  $\{I\}$ . It is easy to remark that  $m_I(f) = 1$ , while  $f$  has degree 2.

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