Zeros of the Riemann Zeta-Function on the Critical Line

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It was shown by Selberg [3] that the Riemann Zeta-function has at least $cT\log T$ zeros on the critical line up to height T, for some positive absolute constant c. Indeed Selberg's method counts only zeros of odd order, and counts each such zero once only, regardless of its multiplicity. With this in mind we shall write $\hat{\gamma}_i$ for the distinct ordinates of zeros of $\zeta(s)$ on the critical line of odd multiplicity. We shall number the points $\hat{\gamma}_i$ so that $0<\hat{\gamma}_1<\hat{\gamma}_2<\ldots$. The purpose of the present note is to extract a little more from Selberg's argument, by obtaining further information on the distribution of the $\hat{\gamma}_i$. This is given in the following result.

Theorem *For any constant $\mu\in(0,2)$ we have*

$$\sum_{\hat{\gamma}_i\le T}(\hat{\gamma}_{i+1}-\hat{\gamma}_i)^{\mu}\ll_{\mu} T(\log T)^{1-\mu}.$$

In particular, if $f(T)$ is any function which tends to infinity with T, then "almost all" intervals $[T\,,\,T+f(T)(\log T)^{-1}]$ contain a point $\hat{\gamma}_i$.

Clearly this result includes Selberg's. Moreover it is apparent that the second statement of the theorem follows from the first. We also remark that, if one merely sums over ordinates γ_i of the zeros in the usual sense, not restricting to those zeros which are on the critical line, then one has

$$\sum_{\gamma_i\le T}(\gamma_{i+1}-\gamma_i)^{\mu}\ll_{\mu} T(\log T)^{1-\mu}$$

for any $\mu>0$, as was shown by Fujii [1].

In giving the proof of our result we shall refer to the version of Selberg's argument presented by Titchmarsh [4: §§10.9-10.22]. The proof uses a "mollifier"

$$\phi(s)=\sum_{\nu\le X}\beta_{\nu}\nu^{-s},$$

in which the numbers β_ν are defined in terms of the coefficients α_ν in the expansion

$$\zeta(s)^{-1/2} = \sum_{\nu=1}^{\infty} \alpha_\nu \nu^{-s} \quad \sigma > 1.$$

Titchmarsh takes

$$\beta_\nu = \alpha_\nu (1 - \frac{\log \nu}{\log X}),$$

but for our purpose the choice

$$\beta_\nu = \begin{cases} \alpha_\nu, & \nu \le X^{1/2}, \\ 2\alpha_\nu \frac{\log X/\nu}{\log X}, & X^{1/2} \le \nu \le X, \end{cases}$$

is required. One then defines

$$F(t) = \frac{\Xi(t)}{t^2 + \frac{1}{4}} |\phi(\frac{1}{2} + it)|^2 e^{(\frac{1}{4}\pi - \frac{1}{2}\delta)t},$$

where δ is small and positive. In fact we shall take $\delta = T^{-1}$, where $[T, 2T]$ is the interval in which we are looking for zeros. With the above definition of $F(t)$ it follows (Titchmarsh [4: Lemma 10.17]) that

$$\int_{-\infty}^{\infty} \left| \int_t^{t+h} F(u) du \right|^2 dt \ll \frac{h}{\delta^{1/2} \log X}. \tag{1}$$

This is subject to the conditions $X = \delta^{-c}$ and $h = (a \log X)^{-1}$, where a, c are positive and satisfy $(a+2)c \le \frac{1}{4}$. Titchmarsh takes a to be constant, but this is unnecessary. We shall set $c = \frac{1}{16}$ so that any value $a \in (0, 2)$ is permissable. We see from (1) that

$$\int_T^{2T} \left| \int_{t-h}^{t+h} F(u) du \right|^2 dt \ll \frac{hT^{1/2}}{\log T}, \tag{2}$$

on changing h into $2h$ and substituting $t - h$ for t. The bound (2) is subject to the conditions $X = T^{1/16}$ and $h = (a' \log T)^{-1}$, where $a' \in (0, \frac{1}{4})$.

The proof of (1), and hence of (2), depends on the definition of β_ν, so we must check that our modification does not materially alter the estimates. It is only Lemma 10.12 of Titchmarsh [4] which needs any change. It is shown that

$$\sum_{\kappa \le X/d} \alpha_\kappa \kappa^{\theta-1} \log \frac{X}{d\kappa} \ll (\frac{X}{d})^\theta (\log \frac{X}{d})^{1/2} \prod_{p|\rho} (1 + p^{-1})^{1/2} \tag{3}$$

uniformly for $0 < \theta \le \frac{1}{2}$, where κ is restricted to integers coprime to ρ. We shall require a corresponding estimate in which the function

$$f(X,d,\kappa) = \begin{cases} \log X/d\kappa, & d\kappa \le X, \\ 0, & d\kappa \ge X, \end{cases}$$

on the left is replaced by

$$g(X,d,\kappa) = \begin{cases} \log X, & d\kappa \le X^{1/2}, \\ 2\log X/d\kappa, & X^{1/2} \le d\kappa \le X, \\ 0, & d\kappa \ge X. \end{cases}$$

However, since

$$g(X,d,\kappa) = 2f(X,d,\kappa) - 2f(X^{1/2},d,\kappa),$$

one sees that (3) remains true with g in place of f.

We shall also require the estimate

$$\int_{-\infty}^{\infty} |F(t)|^2 dt \ll \frac{\log 1/\delta}{\delta^{1/2}\log X} \ll T^{1/2} \tag{4}$$

given by Lemma 10.18 of Titchmarsh [4]. The proof of this requires no modification.

We now establish a lower bound for

$$\int_{t-h}^{t+h} |F(u)|du = J(t),$$

say, on the interval $T \le t \le 2T$. Titchmarsh does this only on average, while we shall, in effect, obtain a lower bound for "almost all" t. We begin by choosing a large constant integer K, and writing

$$w(z) = (\frac{\sin z}{z})^{2K},$$

so that

$$\int_{-\infty}^{\infty} e^{i\lambda t} w(t)dt = 0 \quad \text{for} \quad |\lambda| \ge 2K.$$

We now consider the integral

$$\int_{1/2-i\infty}^{1/2+i\infty} \zeta(s+it)\phi(s+it)^2 w(\frac{s-\frac{1}{2}}{i\Delta})ds = I, \tag{5}$$

say, where

$$\frac{1}{\log T} \le \Delta \le T^{3/4}. \tag{6}$$

The integral will converge if K is chosen large enough. We now move the line of integration to $\sigma = 2$, producing a residue

$$\phi(1)^2 w(\frac{\frac{1}{2} - it}{i\Delta}) \ll (\log T)^2 (\frac{e^{1/2\Delta}}{|t|/\Delta})^{2K} \ll (\log T)^{-2}.$$

On the line $\sigma = 2$ we may integrate termwise. We have

$$\int_{2-i\infty}^{2+i\infty} n^{-s} w(\frac{s - \frac{1}{2}}{i\Delta}) ds = i\Delta \int_{-\infty}^{\infty} n^{-1/2-it\Delta} w(t) dt$$

on moving the line of integration back to $\sigma = 1/2$, so that terms for which $\Delta \log n \geq 2K$ make no contribution. Since

$$\zeta(s)\phi(s)^2 = \sum_{n=1}^{\infty} a_n n^{-s}$$

with $a_1 = 1$ and $a_n = 0$ for $2 \leq n \leq X^{1/2}$, we now see that

$$I = i\Delta C_K + O(\frac{1}{\log^2 T}),$$

where

$$C_K = \int_{-\infty}^{\infty} w(t) dt > 0,$$

providing that

$$\Delta \geq \frac{64K}{\log T}. \tag{7}$$

At this point we observe that if $h \leq T^{3/4}$ then

$$\begin{aligned} T^{1/4} J(t) &\gg \int_{t-h}^{t+h} |\zeta(\frac{1}{2} + iu)\phi(\frac{1}{2} + iu)^2| du \\ &\geq \int_{t-h}^{t+h} |\zeta(\frac{1}{2} + iu)\phi(\frac{1}{2} + iu)^2| w(\frac{u}{\Delta}) du \\ &\geq I + O\{\int_{|u|\geq h} |\zeta(\frac{1}{2} + i(u+t))\phi(\frac{1}{2} + i(u+t))^2| \frac{du}{(|u|/\Delta)^{2K}}\}, \end{aligned}$$

whence

$$T^{1/4} J(t) + \int_{|u|\geq h} |\zeta(\frac{1}{2} + i(u+t))\phi(\frac{1}{2} + i(u+t))^2| \frac{du}{(|u|/\Delta)^{2K}} \gg \Delta.$$

Since

$$\zeta(\frac{1}{2} + i(u+t))\phi(\frac{1}{2} + i(u+t))^2 \ll (T + |u|)^{1/4} X$$

for $T \leq t \leq 2T$, it follows that the range $|u| \geq T/2$ will contribute only $O(\Delta/T)$, say. Here we use the facts that $\Delta \leq T^{3/4}$, by (6), and that K is sufficiently large. Moreover

$$\begin{aligned}\int_{h\leq|u|\leq T/2} &|\zeta(\frac{1}{2}+i(u+t))\phi(\frac{1}{2}+i(u+t))^2|(|u|/\Delta)^{-2K}du \\ &\ll \quad T^{1/4}\int_{h\leq|u|\leq T/2}|F(t+u)|(|u|/\Delta)^{-2K}du \\ &= \quad T^{1/4}K(t),\end{aligned}$$

say. It follows that

$$J(t)+K(t) \gg T^{-1/4}\Delta. \tag{8}$$

We now observe that

$$\int_T^{2T} K(t)dt = \int_{h\leq|u|\leq T/2}(|u|/\Delta)^{-2K}\left\{\int_{T+u}^{2T+u}|F(v)|dv\right\}du,$$

and Cauchy's inequality, in conjunction with (4) yields

$$\int_{T+u}^{2T+u}|F(v)|dv \ll T^{1/2}\{\int_{T/2}^{5T/2}|F(v)|^2dv\}^{1/2} \ll T^{3/4}.$$

We therefore see that

$$\int_T^{2T} K(t)dt \ll \frac{hT^{3/4}}{(h/\Delta)^{2K}}, \tag{9}$$

since

$$\int_{h\leq|u|\leq T/2}(|u|/\Delta)^{-2K}du \ll \frac{h}{(h/\Delta)^{2K}}.$$

We shall write (8) as

$$J(t)+K(t) \geq CT^{-1/4}\Delta,$$

and define

$$R_h = \{t\in[T,2T] : J(t) \leq \frac{C}{2}T^{-1/4}\Delta\}.$$

Then $K(t) \gg T^{-1/4}\Delta$ on R_h, whence

$$T^{-1/4}\Delta\mathrm{mes}(R_h) \ll \frac{hT^{3/4}}{(h/\Delta)^{2K}},$$

by (9). It follows that

$$\mathrm{mes}(R_h) \ll \frac{T}{(h/\Delta)^{2K-1}}. \tag{10}$$

We can now complete the proof of the theorem. We consider the set S_h of $t \in [T, 2T]$ for which the interval $t - h \leq u \leq t + h$ contains no sign change of the function $F(u)$, or equivalently no zero of odd order of $\zeta(\frac{1}{2} + iu)$. Thus

$$J(t) = |\int_{t-h}^{t+h} F(u)du|$$

for $t \in S_h$. If $t \in S_h \setminus R_h$ then $J(t) \geq \frac{C}{2}T^{-1/4}\Delta$, whence (2) yields

$$(\frac{C}{2}T^{-1/4}\Delta)^2 \mathrm{mes}(S_h \setminus R_h) \ll \frac{hT^{1/2}}{\log T}.$$

Thus

$$\mathrm{mes}(S_h \setminus R_h) \ll \frac{hT}{\Delta^2 \log T}.$$

On choosing

$$\Delta = h(h \log T)^{-1/(2K+1)}$$

we therefore deduce from (10) that

$$\mathrm{mes}(S_h) \ll T(h \log T)^{-1+2/(2K+1)}.$$

Our choice of Δ will satisfy the conditions (6) and (7) if $h = (a' \log T)^{-1}$ with $h \leq T^{3/4}$ and $0 < a' \leq a'(K)$, say.

We are now ready to estimate

$$\sum_{\hat{\gamma}_i} (\hat{\gamma}_{i+1} - \hat{\gamma}_i)^{\mu}$$

for $T \leq \hat{\gamma}_i \leq 2T$. We shall choose K to be a fixed integer such that

$$\frac{2}{2K+1} < 2 - \mu.$$

According to a result of Hardy and Littlewood [2] we have $\hat{\gamma}_{i+1} - \hat{\gamma}_i \ll T^{\theta}$ for any $\theta > \frac{1}{4}$, so that it suffices to prove that

$$\sum_{\hat{\gamma}_i} (\hat{\gamma}_{i+1} - \hat{\gamma}_i)^{\mu} \ll T(\log T)^{1-\mu} \tag{11}$$

for $T \leq \hat{\gamma}_i < \hat{\gamma}_{i+1} \leq 2T$. Moreover, summands for which

$$\hat{\gamma}_{i+1} - \hat{\gamma}_i \leq \frac{8}{a'(K) \log T}$$

clearly make a satisfactory contribution. We shall classify the remaining terms according to the value of $h = 2^H, \; H = 1, 2, \ldots$, for which

$$4h < \hat{\gamma}_{i+1} - \hat{\gamma}_i \leq 8h.$$

Thus we may assume that

$$\frac{1}{a'(K)\log T} \le h \le T^{3/4}.$$

Since $t \in S_h$ for $\hat{\gamma}_i + h \le t \le \hat{\gamma}_i + 2h$, we see that the number of points $\hat{\gamma}_i$ corresponding to any such h is at most

$$h^{-1}\text{mes}(S_h) \ll h^{-2+2/(2K+1)}T(\log T)^{-1+2/(2K+1)}.$$

The corresponding contribution to (11) is therefore

$$\ll h^{\mu-2+2/(2K+1)}T(\log T)^{-1+2/(2K+1)},$$

and summing over $h = 2^H \gg (\log T)^{-1}$ yields the required result.

References

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