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# ZEROS OF THE ZAK TRANSFORM ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. Let $G$ be a locally compact abelian group. The notion of Zak transform on $L^2(\mathbb{R}^d)$ extends to $L^2(G)$. Suppose that $G$ is compactly generated and its connected component of the identity is non-compact. Generalizing a classical result for $L^2(\mathbb{R})$, we then prove that if $f \in L^2(G)$ is such that its Zak transform $Zf$ is continuous on $G \times \widehat{G}$, then $Zf$ has a zero.

## 1. Introduction

The Zak transform on the real line, sometimes also referred to as the Weil-Brezin map, was introduced in 1967 by Zak [11] to construct a quantum mechanical representation for the description of the motion of a Bloch electron in the presence of a magnetic or electric field. Subsequently it proved to be an important tool in applied areas such as signal theory, wavelet analysis and solid state physics (compare the survey article [7] and the references therein).

For $f \in L^2(\mathbb{R})$ the Zak transform $Zf$ is the function on $\mathbb{R} \times \mathbb{R}$ defined by

$$Zf(x,y) = \sum_{k=-\infty}^{\infty} f(x+k)e^{2\pi i y k}.$$

A striking property of the Zak transform, independently shown by Zak [3] and Janssen [6], is that $Zf$ has a zero whenever $Zf$ is continuous on $\mathbb{R} \times \mathbb{R}$. Actually, in certain special cases like when $f$ is the Gaussian, this follows from elementary properties of theta series.

Now, the notion of the Zak transform admits a natural generalization to locally compact abelian groups (see Section 3). Given a locally compact abelian group $G$, its dual group $\widehat{G}$ and a uniform lattice $K$ in $G$, the Zak transform, associated to $K$, of $f \in L^2(G)$ can be defined (almost everywhere) on $G \times \widehat{G}$ by

$$Zf(x,\omega) = \sum_{k \in K} f(xk)\omega(k).$$

The main purpose of this note is to extend the above result to compactly generated locally compact abelian groups. In fact, we are going to establish the following stronger result.

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**Theorem.** *Let $G$ be a compactly generated locally compact abelian group with non-compact connected component of the identity, $K$ a uniform lattice in $G$ and $\Gamma$ the annihilator of $K$ in the dual group $\widehat{G}$ of $G$. Suppose that $g : G \times \widehat{G} \to \mathbb{C}$ is a continuous function satisfying the quasi-periodicity relation*

$$g(xk, \omega\gamma) = \overline{\omega(k)} g(x, \omega)$$

*for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$. Then $g$ has a zero.*

The converse to the theorem also holds (Remark 1 in Section 4). As an immediate consequence of the theorem we obtain the following:

**Corollary.** *Let $G$ and $K$ be as in the theorem and $Z$ the associated Zak transform. Let $f \in L^2(G)$ and suppose that $Zf$ is continuous on $G \times \widehat{G}$. Then $Zf$ has a zero.*

It is worthwhile to point out that conversely the corollary implies the theorem at least when $G$ is first countable (compare Remark 5).

The proof of the theorem will be given in Section 2, whereas in Section 3 we deal with the Zak transform. Finally, in Section 4 we conclude with some remarks.

## 2. Proof of the theorem

Let $G$ be an arbitrary locally compact abelian group and let $K$ be a *uniform lattice* in $G$, that is, a discrete subgroup of $G$ with compact quotient group $G/K$. In the sequel, $\Gamma$ will denote the annihilator of $K$ in $\widehat{G}$,

$$\Gamma = A(K, \widehat{G}) = \{\gamma \in \widehat{G} : \gamma(k) = 1 \text{ for all } k \in K\}.$$

Then $\Gamma$ is a uniform lattice in $\widehat{G}$ since $\Gamma$ is topologically isomorphic to $\widehat{G/K}$ and $\widehat{G}/\Gamma$ is topologically isomorphic to $\widehat{K}$ (via the restriction map $\omega\Gamma \to \omega|K$). The following lemma is required in the proof of the theorem.

**Lemma 1.** *Let $\mathcal{H}$ be a downward directed system of compact subgroups of $G$ (with normalized Haar measures) such that $\bigcap_{H \in \mathcal{H}} H = \{e\}$. Let $g$ be a continuous function on $G \times \widehat{G}$ such that*

$$g(xk, \omega\gamma) = \overline{\omega(k)} g(x, \omega)$$

*for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$. For each $H \in \mathcal{H}$, define $g_H$ on $G \times \widehat{G}$ by*

$$g_H(x, \omega) = \int_H g(xh, \omega) dh.$$

*Then $g_H$ is continuous and satisfies $g_H(xk, \omega\gamma) = \overline{\omega(k)} g_H(x, \omega)$. If every $g_H$ has a zero, then $g$ has a zero.*

*Proof.* That $g_H$ is continuous follows immediately from the uniform continuity of $g$ on compact subsets of $G \times \widehat{G}$. Moreover, for $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$,

$$g_H(xk, \omega\gamma) = \int_H g(xkh, \omega) dh = \overline{\omega(k)} \int_H g(xh, \omega) dh = \overline{\omega(k)} g_H(x, \omega).$$

Now suppose that every $g_H$ has a zero. Since $G/K$ and $\widehat{G}/\Gamma$ are compact, there exist compact subsets $C$ of $G$ and $\Delta$ of $\widehat{G}$ such that $G = CK$ and $\widehat{G} = \Delta\Gamma$. Due to the quasi-periodicity, for each $H \in \mathcal{H}$ there exist $x_H \in C$ and $\omega_H \in \Delta$ such that $g_H(x_H, \omega_H) = 0$. $C$ and $\Delta$ being compact, by passing to a subnet if necessary, we can assume that $x_H \to x$ and $\omega_H \to \omega$ for some $x \in C$ and $\omega \in \Delta$. Finally, employing the uniform continuity of $g$ on compact sets once more, we obtain that

$$|g(x,\omega)| = |g(x,\omega) - g_H(x_H,\omega_H)|$$

$$= \Big| \int_H \Big( g(x,\omega) - g(x_H h,\omega_H) \Big) dh \Big| \le \int_H |g(x,\omega) - g(x_H h,\omega_H)| dh,$$

which converges to zero as $H \to \{e\}$. □

We now turn to the proof of the theorem. Notice first that by the structure theorem for compactly generated locally compact abelian groups [5, Theorem 9.8], $G$ is of the form $G = \mathbb{R}^p \times \mathbb{Z}^q \times C$ where $C$ is a compact group and $p \ge 1$ since by hypothesis $G$ has a non-compact connected component of the identity. Now, compact groups are projective limits of Lie groups [10, p.99]. Therefore, there exists a system $\mathcal{H}$ of closed subgroups $H$ of $C$ as in Lemma 1 such that $C/H$ is a Lie group for every $H \in \mathcal{H}$. Thus, for each $H \in \mathcal{H}$, there is a closed subgroup $L_H$ of $C$ such that $H \subseteq L_H, L_H$ is of finite index in $C$ and $L_H/H = \mathbb{T}^{r_H}$ for some $r_H \in \mathbb{N}_0$.

By Lemma 1, for any such $H, g_H$ is continuous, and once we have established that $g_H$ has a zero on $G \times \widehat{G}$, it follows that $g$ has a zero as well. To that end, fix $H$ and set $L = L_H$ and $r = r_H$. Replacing $g_H$ by $g$, we can therefore assume that $g$ is constant on cosets of $H$. Let $\pi : G \to G/H$ denote the quotient homomorphism. Then $\pi(K) = KH/H$ is a uniform lattice in $G/H$, and

$$A(\pi(K), \widehat{G/H}) = \{\chi \in \widehat{G/H} : \chi \circ \pi \in A(K, \widehat{G})\}.$$

Now, the function $\tilde{g} : G/H \times \widehat{G/H} \to \mathbb{C}$ defined by

$$\tilde{g}(\pi(x), \chi) = g(x, \chi \circ \pi),$$

$x \in G$, $\chi \in \widehat{G/H}$, is continuous and satisfies the equation

$$\tilde{g}(\pi(x)\pi(k), \chi\,\delta) = \tilde{g}(\pi(x), \chi)\overline{(\chi \circ \pi)(k)}$$

for all $x \in G$, $k \in K$, $\chi \in \widehat{G/H}$ and $\delta \in A(\pi(K), \widehat{G/H})$. It suffices to show that $\tilde{g}$ has a zero. Thus, after moving to $G/H$, we can assume that $L = \mathbb{T}^r$. Towards a contradiction, suppose that $g(x,\omega) \neq 0$ for all $(x,\omega) \in G \times \widehat{G}$.

In what follows, for $x \in G$ and $\omega \in \widehat{G}$, let $x_1 \in \mathbb{R}^p$ and $\omega_1 \in \widehat{\mathbb{R}^p}$ denote the first component of $x$ and $\omega$, respectively. When convenient, we shall identify $\mathbb{R}^p$ with $\widehat{\mathbb{R}^p}$ by writing $\omega_1(x_1) = \exp 2\pi i \langle x_1, \omega_1 \rangle$. Let $1_C$ be the trivial character of $C$ and $e_r : \mathbb{R}^r \to \mathbb{T}^r$ the covering homomorphism given by

$$e_r(u) = (e^{2\pi i u_1}, \dots, e^{2\pi i u_r})$$

for $u = (u_1, \dots, u_r) \in \mathbb{R}^r$. We define homomorphisms

$$\varphi_1 : \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^p \times \{0\} \times \mathbb{T}^r \subseteq G, \quad (x_1, u) \to (x_1, 0, e_r(u))$$

and

$$\varphi_2 : \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q} \to \widehat{\mathbb{R}^p} \times \widehat{\mathbb{Z}^q} \times \{1_C\} \subseteq \widehat{G}, \quad (\omega_1, \chi) \to (\omega_1, \chi|_{\mathbb{Z}^q}, 1_C).$$

Since $g$ is continuous and has no zero on $G \times \widehat{G}$, we can consider the continuous function

$$(x_1, u, \omega_1, \chi) \to \frac{g(\varphi_1(x_1,u), \varphi_2(\omega_1,\chi))}{|g(\varphi_1(x_1,u), \varphi_2(\omega_1,\chi))|}$$

on $S = \mathbb{R}^p \times \mathbb{R}^r \times \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q}$. Since $S$ is simply connected, there exists a continuous function $\varphi : S \to \mathbb{R}$ such that

$$\exp 2\pi i \varphi(x', \omega') = \frac{g(\varphi_1(x'), \varphi_2(\omega'))}{|g(\varphi_1(x'), \varphi_2(\omega'))|}$$

for all $x' \in \mathbb{R}^p \times \mathbb{R}^r$ and $\omega' \in \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q}$. Since $\varphi_1$ and $\varphi_2$ are homomorphisms, the quasi-periodicity relation for $g$ implies that

$$\exp 2\pi i[\varphi(x'k', \omega') - \varphi(x', \omega')] = \overline{\varphi_2(\omega')(\varphi_1(k'))}$$

$$= \overline{\omega_1(k_1)} = \exp(-2\pi i \langle \omega_1, k_1 \rangle)$$

for all $(x', \omega') \in S$ and $k' \in \varphi_1^{-1}(K)$, and

$$\exp 2\pi i[\varphi(x', \omega'\gamma') - \varphi(x', \omega')] = 1$$

for all $(x', \omega') \in S$ and $\gamma' \in \varphi_2^{-1}(\Gamma)$.

Since $S$ is connected and $\varphi$ is continuous, it follows that given $k'$ and $\gamma'$, there are integers $m_1(k')$ and $m_2(\gamma')$ such that

(1) $\varphi(x'k', \omega') - \varphi(x', \omega') + \langle k_1, \omega_1 \rangle = m_1(k')$

and

(2) $\varphi(x', \omega'\gamma') - \varphi(x', \omega') = m_2(\gamma')$

for all $x' \in \mathbb{R}^p \times \mathbb{R}^r$ and $\omega' \in \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q}$. Applying (1) first and then (2) yields

$$\begin{aligned} \varphi(x'k', \omega'\gamma') &= \varphi(x', \omega'\gamma') - \langle k_1, \omega_1 \rangle - \langle k_1, \gamma_1 \rangle + m_1(k') \\ &= \varphi(x', \omega') + m_2(\gamma') - \langle k_1, \omega_1 \rangle - \langle k_1, \gamma_1 \rangle + m_1(k'). \end{aligned}$$

On the other hand, applying (1) and (2) in the reverse order gives

$$\begin{aligned} \varphi(x'k', \omega'\gamma') &= \varphi(x'k', \omega') + m_2(\gamma') \\ &= \varphi(x', \omega') - \langle k_1, \omega_1 \rangle + m_1(k') + m_2(\gamma'). \end{aligned}$$

Subtracting these two equations shows that

$$\langle k_1, \gamma_1 \rangle = 0$$

for all pairs $(k_1, \gamma_1)$ such that $(k_1, 0, k_3) \in K$ for some $k_3 \in \mathbb{T}^r$ and $(\gamma_1, \gamma_2, 1_C) \in \Gamma$ for some $\gamma_2 \in \widehat{\mathbb{Z}^q}$.

We are now going to show that this is impossible. Notice first that, since $G' = \mathbb{R}^p \times L$ is open in $G$, $G'/(G' \cap K)$ is topologically isomorphic to $G'K/K \subseteq G/K$, which is compact. Hence $K \cap G'$ is cocompact in $G'$. Let $K_1$ denote the set of first components of elements in $K \cap G'$. Then $K_1$ contains a vector space basis for $\mathbb{R}^p$. Indeed, otherwise

$$K \cap G' \subseteq K_1 \times L \subseteq V \times L \subseteq \mathbb{R}^p \times L = G'$$

for some proper subspace $V$ of $\mathbb{R}^p$, which contradicts the fact that $G'/(K \cap G')$ is compact.

Thus it only remains to verify that there exist $\gamma_1 \in \widehat{\mathbb{R}^p}$ and $\gamma_2 \in \widehat{\mathbb{Z}^q}$ such that $\gamma_1 \neq 0$ and $(\gamma_1, \gamma_2, 1_C) \in \Gamma$. Assume that $(\gamma_1, \gamma_2, 1_C) \in A(K, \widehat{G})$ only if $\gamma_1 = 0$. Then

$$A(KC, \widehat{G}) \subseteq A(\mathbb{R}^p \times C, \widehat{G}),$$

and hence $KC \supseteq \mathbb{R}^p \times C$, whence

$$K/(K \cap C) = KC/C \supseteq \mathbb{R}^p,$$

which is impossible since $K$ is discrete. This finishes the proof of the theorem.

The idea of writing $g/|g|$, when possible, as the exponential of some continuous function occurs already in the proofs that Zak [3] and Janssen [6] gave for the existence of a zero in the case $G = \mathbb{R}$. For a different proof compare [1, p.18].

## 3. The Zak transform and zeros

If $G$ is a locally compact abelian group and $K$ a uniform lattice in $G$, then a *fundamental domain* for $K$ will mean a Borel subset $S$ of $G$ such that every $x \in G$ can be uniquely written in the form $x = sk$ where $s \in S$ and $k \in K$.

Generalizing the classical notion of the Zak transform for the uniform lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$, we are going to introduce the Zak transform on $L^2(G)$ associated to $K$. The first step is to guarantee the existence of a fundamental domain for $K$.

**Lemma 2.** *Let $G$ be a locally compact abelian group and $K$ a uniform lattice in $G$. Then there exists a relatively compact fundamental domain for $K$.*

*Proof.* We assume first that $G$ is compactly generated. Since $G$ is a projective limit of second countable groups [10, p.104] and $K$ is discrete, there exists a compact subgroup $C$ of $G$ such that $C \cap K = \{e\}$ and $G/C$ is second countable. By [8, Lemma 1.1] there exists a relatively compact fundamental domain $Q$ for $KC/C$ in $G/C$. Let $q : G \to G/C$ denote the quotient homomorphism, and set $S = q^{-1}(Q)$. Clearly, $S$ is a relatively compact Borel set, and using the fact that $K \cap C = \{e\}$, it is easy to check that $S$ is indeed a fundamental domain for $K$.

Now, drop the assumption that $G$ is compactly generated and choose an open compactly generated subgroup $H$ of $G$. Since $K \cap H$ is a uniform lattice in $H$, by the preceding paragraph there exists a relatively compact fundamental domain $S$ for $K \cap H$ in $H$. As $H$ is open and $G/K$ is compact, $KH$ has finite index in $G$. Let $F$ be a coset representative system for $KH$ in $G$, and let $T = FS$. Then $T$ is a relatively compact Borel set, and as above it is straightforward to verify that $T$ is a fundamental domain for $K$ in $G$. □

By Lemma 2 there exist relatively compact fundamental domains $S$ for $K$ in $G$ and $\Omega$ for $\Gamma$ in $\widehat{G}$.

Let the Haar measure on $G$ be normalized so that Weil's formula holds, if we take on $G/K$ the normalized Haar measure and the counting measure on $K$. Clearly, if $G$ is $\sigma$-compact (equivalently, $K$ is countable), then $S$ has positive measure ($|S| > 0$). However, this is also true in the general case. To see this, choose a compactly generated open subgroup $H$ of $G$ containing $S$ and observe that $Sk \cap H \neq \emptyset$ if and only if $k \in H$. Since $H$ is $\sigma$-compact and $K$ is discrete, there are only countably many such $k$. Thus $H$ is a countable union of sets $Sk$, $k \in K$, whence $|S| > 0$.

The map $\Phi : S \to G/K$, $x \to xK$ is a continuous bijection. For each measurable subset $M$ of $S$ and with $\chi_M$ the characteristic function of $M$, Weil's formula gives

$$|M| = \int_G \chi_M(x)dx = \int_{G/K} \left( \sum_{k \in K} \chi_M(xk) \right) d(xK) = |\Phi M|.$$

Hence $\Phi$ maps the measure on $S$ induced by the Haar measure on $G$ to the normalized Haar measure on $G/K$.

Similarly, normalizing the Haar measures on $\widehat{G}$ and $\widehat{G}/\Gamma$ appropriately, the mapping $\Omega \to \widehat{G}/\Gamma$, $\omega \to \omega\Gamma$ transforms the induced measure on $\Omega$ into the Haar measure on $\widehat{G}/\Gamma$, and $|\Omega| = 1$.

The next lemma will show that for an arbitrary locally compact abelian group the Zak transform can be defined as indicated in the introduction.

**Lemma 3.** *Retain the preceding assumptions and notations, and let $f \in L^2(G)$. Then, for almost all $(x, \omega) \in S \times \Omega$,*

$$Zf(x, \omega) = \sum_{k \in K} f(xk)\omega(k)$$

*converges, and the function $Zf$ belongs to $L^2(S \times \Omega)$ and satisfies $\|Zf\|_2 = \|f\|_2$.*

*Proof.* For $k \in K$, define $f_k \in L^2(S \times \Omega)$ by $f_k(x, \omega) = f(xk)\omega(k)$. Then

$$\sum_{k \in K} \|f_k\|_2^2 = \sum_{k \in K} \int_S \int_\Omega |f_k(x, \omega)|^2 d\omega dx = \sum_{k \in K} \int_S |f(xk)|^2 dx = \|f\|_2^2.$$

We claim that $\langle f_k, f_l \rangle = 0$ for $k, l \in K, k \neq l$. To show this recall that if $C$ is a compact abelian group and $\varphi$ a non-trivial character of $C$, then $\int_C \varphi(y)dy = 0$ [5, Lemma 23.19]. Applying this to $C = \widehat{G}/\Gamma$ and the character $\varphi$ defined by

$$\varphi(\omega\Gamma) = \omega(kl^{-1}), \quad \omega \in \widehat{G},$$

we obtain

$$\int_\Omega \omega(kl^{-1}) d\omega = \int_{\widehat{G}/\Gamma} \varphi(\omega\Gamma) d(\omega\Gamma) = 0,$$

and this in turn implies

$$\langle f_k, f_l \rangle = \int_S \int_\Omega f(xk)\overline{f(xl)}\omega(kl^{-1}) d\omega dx = 0.$$

It follows that the series $\sum_{k \in K} f_k$ converges in $L^2(S \times \Omega)$ and satisfies

$$\|\sum_{k \in K} f_k\|_2^2 = \sum_{k \in K} \|f_k\|_2^2 = \|f\|_2^2.$$

In particular, $Zf(x, \omega)$ exists for almost all $(x, \omega) \in S \times \Omega$. □

We can now define the Zak transform $Zf$ for $f \in L^2(G)$. Notice first that for every $(k, \gamma) \in K \times \Gamma$ and any finite subset $H$ of $K$,

$$\sum_{h \in H} f(xkh)(\omega\gamma)(h) = \overline{\omega(k)} \sum_{l \in H} f(xl)\omega(l).$$

Thus $Zf(xk, \omega\gamma)$ converges if and only if $Zf(x, \omega)$ does. It follows from Lemma 3 that

$$Zf(x, \omega) = \sum_{k \in K} f(xk)\omega(k)$$

is defined for locally almost all $(x, \omega) \in G \times \widehat{G}$ (and, in fact, for a.a. $(x, \omega)$ if $G$ is $\sigma$-compact), and this function is called the *Zak transform* of $f$. We say that $Zf$ *is continuous on* $G \times \widehat{G}$ if there exists a continuous function $g$ on $G \times \widehat{G}$ which agrees with $Zf$ locally a.e. on $G \times \widehat{G}$. Of course, such a function $g$ then satisfies the quasi-periodicity relation $g(xk, \omega\gamma) = \overline{\omega(k)} g(x, \omega)$ for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$. Hence an application of the theorem yields the corollary.

## 4. Some remarks

We finish the paper by adding remarks concerning the hypotheses of the theorem and some application.

*Remark* 1. The converse to the theorem also holds. That is, if $G$ is a compactly generated locally compact abelian group, then $G_0$, the connected component of the identity, must be non-compact provided that $G$ has the following property: For every uniform lattice $K$ in $G$ and $f \in L^2(G)$, $Zf$ has a zero whenever $Zf$ is continuous.

In fact, suppose that $G_0$ is compact so that $G = D \times C$ where $D$ is discrete and $C$ is compact. Choosing $K = D$ and $f = \chi_C$, one obtains for $x = dc$, $d \in D$, $c \in C$, and $\omega \in \widehat{G}$,

$$Zf(x, \omega) = \sum_{k \in D} f(xk)\omega(k) = \overline{\omega(d)}.$$

This formula shows that $Zf$ is continuous and of modulus 1.

*Remark* 2. In general, a locally compact abelian group $G$ need not contain a uniform lattice. The following example was kindly communicated by the referee.

Suppose $G$ is the group $(\mathbb{Z}_4)^\infty$ with the topology obtained when the subgroup $C$ generated by all elements of order 2 is declared to be open and compact. Then every discrete subgroup $K$ of $G$ has to be finite. Indeed, $K \cap C$ is finite and $x \to x^2$ is a homomorphism from $K$ into $K \cap C$ with kernel $K \cap C$.

However, if $G$ is of the form $G = \mathbb{R}^p \times D \times C$, where $D$ is discrete and $C$ is compact, then we can take $K = \mathbb{Z}^p \times D$. More specifically, if $G$ is compactly generated, say $G = \mathbb{R}^p \times \mathbb{Z}^q \times C$, then an abundance of uniform lattices can be constructed as follows. Let $h_1$ be a homomorphism of $\mathbb{Z}^p \subseteq \mathbb{R}^p$ into $C$ and let $h_2$ and $h_3$ be homomorphisms of $\mathbb{Z}^q$ into $\mathbb{R}^p$ and $C$, respectively. Then

$$K = \{(x_1 + h_2(x_2), x_2, h_1(x_1) + h_3(x_2)) : x_1 \in \mathbb{Z}^p, x_2 \in \mathbb{Z}^q\}$$

is a uniform lattice in $G$.

*Remark* 3. The condition that $Zf$ be continuous is satisfied whenever $f$ is continuous and rapidly decreasing outside of compact subsets of $G$. More precisely, it is well-known that if $f$ is a continuous function on $\mathbb{R}^d$ such that $|f(x)| \le c(1+\|x\|_2)^{-\alpha}$ for some $\alpha > 1$ and $c > 0$, then $Zf$ is continuous. Slightly more general, it is not difficult to see that for $G = \mathbb{R}^p \times \mathbb{Z}^q \times C \subseteq \mathbb{R}^p \times \mathbb{R}^q \times C$, a similar hypothesis with respect to the $\mathbb{R}^p$ and $\mathbb{R}^q$ variables is sufficient.

For the two final remarks, let $K$ denote a uniform lattice in the locally compact abelian group $G$, $\Gamma$ the annihilator of $K$ in $\widehat{G}$ and $Z$ the Zak transform associated to $K$.

*Remark* 4. Let $S$ and $\Omega$ be relatively compact fundamental domains for $K$ in $G$ and $\Gamma$ in $\widehat{G}$, respectively. We have seen (Lemma 3) that, after suitably normalizing Haar measures, $Z$ maps $L^2(G)$ unitarily into $L^2(S \times \Omega)$. It can be shown that $Z$ is surjective provided that the mappings $S \to G/K$ and $\Omega \to \widehat{G}/\Gamma$ induce Hilbert space isomorphisms $L^2(S) \to L^2(G/K)$ and $L^2(\Omega) \to L^2(\widehat{G}/\Gamma)$, (compare the proof for $G = \mathbb{R}^d$ in [2]). This latter condition is satisfied if $S \to G/K$ and $\Omega \to \widehat{G}/\Gamma$ are Borel isomorphisms, that is, if $S$ and $\Omega$ arise from Borel cross-sections $G/K \to G$

and $\widehat{G}/\Gamma \to \widehat{G}$. Now, the existence of such cross-sections is guaranteed when $G$ (and hence $\widehat{G}$) is second countable ([8, Lemma 1.1] and [9, Theorem 4.2]).

*Remark* 5. Let $G$ be a first countable compactly generated locally compact abelian group, and let $K$ be a uniform lattice in $G$ and $\Gamma$ the annihilator of $K$ in $\widehat{G}$. Choose relatively compact Borel sets $S$ in $G$ and $\Omega$ in $\widehat{G}$ such that the quotient mappings are Borel isomorphisms (see Remark 4). Suppose that $g$ is a continuous function on $G \times \widehat{G}$ satisfying the quasi-periodicity relation $g(xk, \omega\gamma) = \overline{\omega(k)}\, g(x, \omega)$ for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$. Then, since $Z : L^2(G) \to L^2(S \times \Omega)$ is surjective, there exists $f \in L^2(G)$ such that $Zf = g$ a.e. on $S \times \Omega$, hence a.e. on $G \times \widehat{G}$. Thus, in this situation, the theorem and the corollary are equivalent.

*Remark* 6. Let $f \in L^2(G)$, and for $k \in K$ and $\gamma \in \Gamma$ define $\varphi_{k,\gamma} \in L^2(G)$ by $\varphi_{k,\gamma}(x) = \gamma(x) f(k^{-1}x)$. The collection of all these functions is called the *Gabor system associated with* $f$. In the classical situation, $G = \mathbb{R}^d$, the question of when this Gabor system forms a frame (an exact frame, an orthonormal basis) for $L^2(G)$ has been a matter of great interest.

In this context the Zak transform plays an important role. For instance, the set $\{\varphi_{k,\gamma} : k \in K, \gamma \in \Gamma\}$ constitutes a frame for $L^2(\mathbb{R}^d)$ with frame bounds $A$ and $B$ precisely when $A \leq |Zf| \leq B$ almost everywhere on $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ (see [2, Theorem 3.16]). Now, the proofs of these results carry over, in a straightforward manner, to a general locally compact abelian group $G$ provided that the mapping $Z : L^2(G) \to L^2(S \times \Omega)$ is onto. By the preceding remark we know this to be true for suitable $S$ and $\Omega$ at least when $G$ is second countable.

In particular, from the corollary we can draw the following conclusion. Suppose that $G$ is a first countable compactly generated locally compact abelian group with non-compact connected component of the identity. If $f \in L^2(G)$ is such that $Zf$ is continuous, then the functions $\varphi_{k,\gamma}$ do not form a frame for $L^2(G)$.

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